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A FINITENESS THEOREM FOR TRANSITIVE FOLIATIONS AND FLAT VECTOR BUNDLES

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ABSTRACT

In this paper we prove a finiteness theorem for the spectral sequence $(E_i(\nabla), (d_\nabla)_i)$ associated to a transitive foliation $\mathcal{F}$ on a compact manifold M, and to a flat vector bundle E over M with flat connection ∇. We also compute some examples of homogeneous Lie foliations on compact connected homogeneous spaces.

1. Introduction

First, recall that for a smooth foliation $\mathcal{F}$ on a smooth manifold M, the spectral sequence $(E_i, d_i) = (E_i(\mathcal{F}), d_i)$ associated to $\mathcal{F}$ arises from the filtered de Rham complex $(A(M), d)$ of $(M, \mathcal{F})$, and converges to the real cohomology $H(M)$ of M (see for example [27, 17, 29]). It is clear that $(E_1^{\cdot,0}, d_1)$ and $E_2^{\cdot,0}$ are respectively the complex $(A_b(M), d)$ of basic forms and the basic cohomology $H_b(M)$ of $\mathcal{F}$. K. S. Sarkaria [27] has proved that E_2 is finite dimensional if $\mathcal{F}$ is transitive and M compact. This result has been used in [29, 1, 19] to prove that E_2 is finite dimensional when $\mathcal{F}$ is Riemannian and M compact. On the other hand, also with this hypothesis, the finite dimensional character and duality of $H_b(M)$ have been studied in [16, 17, 28, 11, 18, 31, 23, 3, 19].

Let $\mathcal{F}$ be a smooth foliation of dimension p and codimension q on a smooth manifold M. Let E be a flat vector bundle over M with flat connection ∇. Then the usual filtration of $A(M)$ induces a filtration in the complex $(A(M,E), d_\nabla)$ of

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smooth forms on M with values in E. With this decreasing filtration, $(A(M,E),d_\nabla)$ is a filtered complex and we have the corresponding spectral sequence $(E_i(\nabla),(d_\nabla)_i)$ associated to $\mathcal{F}$ and E, which collapses at the $(q+1)$-th term and converges to the real cohomology $H(M,E)$ of $(A(M,E),d_\nabla)$. On the other hand, we consider in $A(M,E)$ the usual $\mathcal{C}^\infty$-topology, turning $(A(M,E),d_\nabla)$ into a Fréchet topological complex. Each $E_i(\nabla)$ has the induced topology and $(d_\nabla)_i$ is continuous. $E_1(\nabla)$ in general is not Hausdorff obtaining two new topological complexes, the closure $\bar{O}$ of the trivial subspace of $E_1(\nabla)$ and the reduction $\mathbb{E}_1(\nabla) = E_1(\nabla)/\bar{O}$ of $E_1(\nabla)$. We shall denote by $\mathbb{E}_2(\nabla)$ the cohomology $H(\mathbb{E}_1(\nabla),(d_\nabla)_1)$.

A case of particular interest is the one where $E = \mathcal{C}(\mathcal{F})$ is the flat vector bundle associated to the Molino commuting sheaf [20, 22] of a transitive foliation $\mathcal{F}$ on a compact connected manifold M. Another particular case is the following. Consider a closed one-form $\gamma \in A^1(M)$ and let ∇ be the flat connection on the trivial vector bundle $E = M \times \mathbb{R}$ with connection form γ with respect to the smooth section σ of E given by $\sigma(x) = (x,1)$. Then the spectral sequence $(E_i(\gamma),(d_\gamma)_i) = (E_i(\nabla),(d_\nabla)_i)$ associated to $\mathcal{F}$ and γ arises from the filtered complex $(A(M),d_\gamma) = (A(M,E),d_\nabla)$, where $d_\gamma = d + \gamma\wedge$. Note that each $E_i(\gamma)$ depends only on the class $[\gamma] \in H^1(M)$, and that $[\gamma] = 0$ if and only if $E_i(\gamma) = E_i$ for all i.

In this paper we study the spectral sequence $(E_i(\nabla),(d_\nabla)_i)$, and using the Riesz theory of compact operators we prove that for a transitive foliation $\mathcal{F}$ on a compact manifold M and a flat vector bundle $E \to M$ with flat connection ∇, the cohomologies $E_2(\nabla)$ and $\mathbb{E}_2(\nabla)$ are finite dimensional Hausdorff and $E_2(\nabla) \cong \mathbb{E}_2(\nabla)$ canonically. We also compute some examples of homogeneous Lie foliations on compact connected homogeneous spaces.

The paper is structured as follows. In Section 2, using the techniques of [27] we construct a compact operator and a parametrix for the complex $(A(M,E),d_\nabla)$ of smooth forms on a compact manifold M with values in a flat vector bundle $E \to M$ with flat connection ∇. In Section 3 the results of Section 2 are applied to the case where M is equipped with a transitive foliation $\mathcal{F}$. Section 4 is devoted to the study of the spectral sequence $(E_i(\nabla),(d_\nabla)_i)$ of the filtered complex $(A(M,E),d_\nabla)$ associated to a smooth foliation $\mathcal{F}$ on a smooth manifold M, and to a flat vector bundle E over M. In Section 5, using the results of Section 3 and the Riesz theory of compact operators [13, 25], we prove that for a transitive foliation $\mathcal{F}$ on a compact manifold M and a flat vector bundle E over M, the cohomologies $E_2(\nabla)$ and $\mathbb{E}_2(\nabla)$ are finite dimensional Hausdorff and $E_2(\nabla) \cong \mathbb{E}_2(\nabla)$ canonically. Finally, in Section 6 we study some examples of homogeneous Lie foliations $\mathcal{F}$ on compact connected homogeneous spaces $M = \Gamma_A \backslash G_A$, and compute the spectral sequence E_i (resp. $E_i(\nabla)$) associated to $\mathcal{F}$ (resp. to $\mathcal{F}$ and the Molino commuting sheaf $\mathcal{C}(\mathcal{F})$).

The results of this paper are applied in [8, 9] to prove a finiteness theorem for Riemannian foliations on compact manifolds, and to show that every Riemannian foliation on a compact manifold is tense (in the sense of [17]). In particular, it follows that the main tautness theorems for Riemannian foliations on compact manifolds, which were proved by several authors, are immediate consequences of our results.

2. Compact operators

In this section, using the techniques of [12, Vol. II] and [27], we construct a compact operator and a parametrix for the complex $A(M, E)$ of smooth forms on a smooth compact manifold M with values in a flat vector bundle E over M.

For any smooth manifold M, TM denotes the tangent bundle of M, $\mathfrak{X}(M) = \Gamma TM$ the Lie algebra of vector fields on M, and $A(M)$ the graded algebra of smooth forms on M. If E is a smooth vector bundle over M, then ΓE denotes the $A^0(M)$-module of smooth sections of E.

Let M be a smooth manifold, and let E be a smooth vector bundle over M. Consider the graded $A(M)$-module

$$\begin{aligned} A(M,E) = \Gamma L(\Lambda TM, E) = \Gamma(\Lambda T^*M \otimes E) &= A(M) \otimes_{A^0(M)} \Gamma E \\ &= \mathrm{Hom}_{A^0(M)}(\Lambda \mathfrak{X}(M), \Gamma E) \end{aligned}$$

of smooth forms on M with values in E. We shall topologise $A(M, E)$ with the usual C^∞-topology turning $A(M, E)$ into a Fréchet topological vector space (in particular, we have $A(M, M \times \mathbb{R}) = A(M)$). Evidently, for any $X \in \mathfrak{X}(M)$, the interior product $i(X)\colon A^r(M, E) \to A^{r-1}(M, E)$ is continuous. Let ∇ be a connection on E. Then the covariant exterior derivative $d_\nabla\colon A^r(M, E) \to A^{r+1}(M, E)$ and the covariant Lie derivative $\theta_\nabla(X)\colon A^r(M, E) \to A^r(M, E)$ for $X \in \mathfrak{X}(M)$ are continuous. Moreover, we have [12, Vol. II]

$$\begin{aligned} &i(X)^2 = 0, i([X,Y]) = [\theta_\nabla(X), i(Y)], \theta_\nabla(X) = i(X)d_\nabla + d_\nabla i(X), \\ &(\theta_\nabla([X,Y]) - [\theta_\nabla(X), \theta_\nabla(Y)])\,\alpha = -R(X,Y) \wedge \alpha, d_\nabla^2 \alpha = R \wedge \alpha, \\ &(\theta_\nabla(X) d_\nabla - d_\nabla \theta_\nabla(X))\,\alpha = i(X)R \wedge \alpha \quad \text{for } X, Y \in \mathfrak{X}(M), \alpha \in A(M,E), \end{aligned}$$

where R is the curvature of ∇. Thus, if the connection ∇ is flat, then

$$\theta_\nabla([X,Y]) = [\theta_\nabla(X), \theta_\nabla(Y)], d_\nabla^2 = 0, \theta_\nabla(X) d_\nabla = d_\nabla \theta_\nabla(X).$$

On the other hand, let E' be a second smooth vector bundle over a smooth manifold M', and let $\tilde{\varphi}$: $E' \to E$ be a smooth bundle map inducing φ: $M' \to M$ and restricting to linear isomorphisms $\tilde{\varphi}_x$: $E'_x \to E_{\varphi(x)}$ in the fibers. Then $\tilde{\varphi}$ induces a continuous linear map $\tilde{\varphi}^{\#}$: $A^r(M,E) \to A^r(M',E')$ given by

$$(\tilde{\varphi}^{\#}\alpha)(x) = \tilde{\varphi}^{\#}_x(\alpha(\varphi(x))), \quad x \in M', \quad \alpha \in A^r(M,E),$$

where $\tilde{\varphi}^{\#}_x$: $L(\Lambda^r T_{\varphi(x)}M, E_{\varphi(x)}) \to L(\Lambda^r T_x M', E'_x)$ denotes the composition of the linear map φ^*: $L(\Lambda^r T_{\varphi(x)}M, E_{\varphi(x)}) \to L(\Lambda^r T_x M', E_{\varphi(x)})$ with the linear isomorphism $(\tilde{\varphi}_x^{-1})_*: L(\Lambda^r T_x M', E_{\varphi(x)}) \to L(\Lambda^r T_x M', E'_x)$. If $Y \in \mathfrak{X}(M')$ and $X \in \mathfrak{X}(M)$ are φ-related, then $\tilde{\varphi}^{\#} \circ i(X) = i(Y) \circ \tilde{\varphi}^{\#}$.

Now, let ∇' be the pullback connection on E' of ∇ along φ. Then $\tilde{\varphi}^{\#} \circ d_\nabla = d_{\nabla'} \circ \tilde{\varphi}^{\#}$. Thus, if $Y \in \mathfrak{X}(M')$ and $X \in \mathfrak{X}(M)$ are φ-related, then $\tilde{\varphi}^{\#} \circ \theta_\nabla(X) = \theta_{\nabla'}(Y) \circ \tilde{\varphi}^{\#}$.

We shall say that a vector space $V \subset \mathfrak{X}(M)$ is **transitive** if the evaluation map e_x: $V \to T_x M$ is surjective for all $x \in M$. According to Section 2.23 in [12, Vol. I], we can always choose a finite dimensional and transitive space $V \subset \mathfrak{X}(M)$.

THEOREM 2.1: *Let M be a compact manifold. Let E be a smooth vector bundle over M and ∇ a connection on E. Then there exist two continuous linear maps s, h: $A(M,E) \to A(M,E)$ of degrees 0 and -1 respectively, such that*

(i) *s is a compact operator;*

(ii) *if ∇ is flat, then $1 - s = d_\nabla h + h d_\nabla$.*

Proof: For all $X \in \mathfrak{X}(M)$, denote by $\tilde{X} \in \mathfrak{X}(E)$ the unique horizontal lift of X with respect to ∇. Let X_t, $t \in \mathbb{R}$ (resp. $\tilde{X}_t$, $t \in \mathbb{R}$) be the flow of the vector field $X \in \mathfrak{X}(M)$ (resp. of the horizontal lift $\tilde{X} \in \mathfrak{X}(E)$ of X). Then, for all $t \in \mathbb{R}$, $\tilde{X}_t$: $E \to E$ is an isomorphism of vector bundles inducing X_t: $M \to M$. Denote by $X_t^{\#}$: $A(M,E) \to A(M,E)$ the continuous linear isomorphism induced by $\tilde{X}_t$.

Now, consider a finite dimensional transitive space $V \subset \mathfrak{X}(M)$ and choose a Riemannian metric g on V. Let $|g|$ be the volume element, and let f be a smooth nonnegative function on V supported in a compact neighbourhood of zero and such that $\int_V f(X) \cdot |g| = 1$. Then we define the continuous linear maps s, h: $A(M,E) \to A(M,E)$ by

$$(2.2)\qquad \begin{aligned} (s\alpha)(x) &= \int_V (X_1^{\#}\alpha)(x) \cdot f(X) \cdot |g| \in L(\Lambda^r T_x M, E_x), \\ (h\alpha)(x) &= -\int_V \int_0^1 (i(X) X_t^{\#}\alpha)(x) \cdot f(X) \cdot dt \cdot |g| \in L(\Lambda^{r-1} T_x M, E_x) \end{aligned}$$

for $x \in M$, $\alpha \in A^r(M,E)$.

Next, from transitivity of V it follows that the evaluation map $e\colon M\times V \to TM$ (given by $(x,X) \mapsto X(x) \in T_xM$) is surjective. Therefore, $N = \operatorname{Ker} e$ is a subbundle of the trivial Riemannian vector bundle $M\times V$ over M. Let $N^{\perp} \cong TM$ be the orthogonal complement of N, and consider the orthogonal projection $\pi\colon M \times V = N \oplus N^{\perp} \to N$. Denote by $\psi\colon M \times V \to N \times M$ the smooth map defined by $(x,X) \mapsto (\pi(x,X), X_1(x))$. It is easy to see that there exists a neighbourhood $U \subset V$ of zero such that ψ is a diffeomorphism on $M \times U$. Choose the function f such that $\operatorname{supp} f \subset U$, and denote by Ω the **volume bundle** $\Omega(M)$ of M. Then by a technique similar to that used in [27] it follows that there exists a smooth section K of the smooth vector bundle

$$L(M \times L(\Lambda^r TM, E), L(\Lambda^r TM, E) \boxtimes \Omega) = L(\Lambda^r TM, E) \boxtimes (L(\Lambda^r TM, E)^* \otimes \Omega)$$

over $M \times M$ such that

$$(s\alpha)(x) = \int_M K(x,y)\alpha(y), \quad x \in M,\ \alpha \in A^r(M,E). \tag{2.3}$$

Hence, $s = s(V,f,g,\nabla)\colon A(M,E) \to A(M,E)$ is a **smoothing operator** (see [5]) with **smooth kernel** K. Thus s is a **compact operator**. This proves (i).

To prove (ii), consider the formula

$$\theta_\nabla(X) = d_\nabla i(X) + i(X) d_\nabla, \quad X \in \mathfrak{X}(M). \tag{2.4}$$

By a direct computation we obtain

$$X_t^{\#}\theta_\nabla(X)\alpha = \left.\frac{dX_s^{\#}\alpha}{ds}\right|_{s=t}, \quad \alpha \in A(M,E),\ X \in \mathfrak{X}(M),\ t \in \mathbb{R}. \tag{2.5}$$

Since X is X_t-related to X for any $X \in \mathfrak{X}(M)$, $t \in \mathbb{R}$, we have

$$i(X) \circ X_t^{\#} = X_t^{\#} \circ i(X), \quad X \in \mathfrak{X}(M),\ t \in \mathbb{R}. \tag{2.6}$$

Now, for each fixed $t \in \mathbb{R}$, let $j_t\colon M \to \mathbb{R} \times M$ be the inclusion map given by $x \mapsto (t,x)$. Then, for $\alpha \in A(\mathbb{R} \times M, \mathbb{R} \times E) \cong A(\mathbb{R} \times M) \otimes_{A^0(M)} \Gamma E$, we obtain

$$d_\nabla \int_0^1 \tilde{j}_t^{\#}\alpha \cdot dt = \int_0^1 d_\nabla \tilde{j}_t^{\#}\alpha \cdot dt. \tag{2.7}$$

Similarly, for each fixed $X \in V$, let $j_X\colon M \to M \times V$ be the inclusion map given by $x \mapsto (x,X)$. Then, for $\alpha \in A(M \times V, E \times V) \cong A(M \times V) \otimes_{A^0(M)} \Gamma E$, we have

$$d_\nabla \int_V \tilde{j}_X^{\#}\alpha \cdot f(X) \cdot |g| = \int_V d_\nabla \tilde{j}_X^{\#}\alpha \cdot f(X) \cdot |g|. \tag{2.8}$$

Finally, suppose that the connection ∇ is flat. Then, for each $X \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$, the automorphism of vector bundles $\tilde{X}_t$: $E \to E$ inducing X_t: $M \to M$ preserves the connection ∇. Hence we have

$$(2.9)\ d_\nabla \circ X_t^\# = X_t^\# \circ d_\nabla, \quad \theta_\nabla(X) \circ X_t^\# = X_t^\# \circ \theta_\nabla(X), \quad X \in \mathfrak{X}(M), t \in \mathbb{R}.$$

Thus if $\alpha \in A(M, E)$, then by (2.2), (2.4), (2.5), (2.6), (2.7), (2.8) and (2.9) it follows that

$$\begin{aligned}(d_\nabla h + h d_\nabla)\alpha &= -\int_V \int_0^1 X_t^\#(d_\nabla i(X) + i(X) d_\nabla)\alpha \cdot f(X) \cdot dt \cdot |g| \\ &= -\int_V \int_0^1 X_t^\# \theta_\nabla(X)\alpha \cdot f(X) \cdot dt \cdot |g| \\ &= -\int_V (X_1^\# \alpha - \alpha) \cdot f(X) \cdot |g| = \alpha - s\alpha. \qquad \blacksquare\end{aligned}$$

Remark: From [5] it follows that the operator s: $A(M, E) \to A(M, E)$ is in fact of **trace class**, the trace being defined by $\operatorname{Tr} s = \int_M \operatorname{Tr} K(x, x)$. Assume now that the vector bundle E over M is flat with flat connection ∇. Denote by $H(M, E)$ the cohomology of the complex $(A(M, E), d_\nabla)$, and by s^r the map s: $A^r(M, E) \to A^r(M, E)$. Then $\sum_r (-1)^r \operatorname{Tr} s^r$ is the Euler characteristic of the finite dimensional cohomology $H(M, E)$. Moreover, Theorem 2.1 proves that h is a **parametrix** for $A(M, E)$.

Example: Let M, V, g and f be as in Theorem 2.1. As usual we will equip the graded algebra $A(M)$ with the exterior derivative d, the interior product $i(X)$ and the Lie derivative $\theta(X)$ for any $X \in \mathfrak{X}(M)$. Consider a one-form $\gamma \in A^1(M)$, and the operators $d_\gamma = d + \gamma \wedge : A^r(M) \to A^{r+1}(M)$, $\theta_\gamma(X) = \theta(X) + i(X)\gamma \cdot : A^r(M) \to A^r(M)$ given by $d_\gamma \alpha = d\alpha + \gamma \wedge \alpha$, $\theta_\gamma(X)\alpha = \theta(X)\alpha + i(X)\gamma \cdot \alpha$ for $\alpha \in A^r(M)$ and $X \in \mathfrak{X}(M)$.

Now, let $E = M \times \mathbb{R}$ be the trivial vector bundle over M, and let ∇ be the connection on E with connection form γ with respect to the smooth section $\sigma \in \Gamma E$ defined by $\sigma(x) = (x, 1)$. Then we have $A(M, E) = A(M)$, $d_\nabla = d_\gamma$ and $\theta_\nabla(X) = \theta_\gamma(X)$, $X \in \mathfrak{X}(M)$. For each $X \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$, let λ_{X_t} be the unique smooth positive function on M determined by $X_t^\# \sigma = \lambda_{X_t} \cdot \sigma$. In particular, $\lambda_{X_0} = \lambda_{0_t} = 1$. It is clear that the functions $(t, x) \mapsto \lambda_{X_t}(x)$ on $\mathbb{R} \times M$ are smooth. Similarly, the function $(t, x, X) \mapsto \lambda_{X_t}(x)$ on $\mathbb{R} \times M \times V$ is smooth. It follows that the continuous linear automorphisms $X_t^\#$: $A(M) \to A(M)$ are given by $\alpha \mapsto X_t^* \alpha \cdot \lambda_{X_t}$. Formula (2.5) implies that

$$(2.10) \qquad i(X) X_t^* \gamma \cdot \lambda_{X_t} = \left. \frac{d\lambda_{X_s}}{ds} \right|_{s=t}.$$

Therefore, from (2.2) it follows that the continuous linear maps $s, h\colon A(M) \to A(M)$ (of degrees 0 and -1 respectively) are given by

$$(2.11) \qquad \begin{aligned} (s\alpha)(x) &= \int_V (X_1^*\alpha)(x) \cdot \lambda_{X_1}(x) \cdot f(X) \cdot |g|, \\ (h\alpha)(x) &= -\int_V \int_0^1 (i(X)X_t^*\alpha)(x) \cdot \lambda_{X_t}(x) \cdot f(X) \cdot dt \cdot |g|. \end{aligned}$$

Moreover, according to Theorem 2.1, there exists a smooth section K of the smooth vector bundle

$$L(M \times \Lambda^r T^*M, \Lambda^r T^*M \boxtimes \Omega) = \Lambda^r T^*M \boxtimes (\Lambda^r TM \otimes \Omega)$$

over $M \times M$ such that

$$(2.12) \qquad (s\alpha)(x) = \int_M K(x,y)\alpha(y).$$

Hence, $s = s(V, f, g, \gamma)\colon A(M) \to A(M)$ is a smoothing operator with smooth kernel K. Thus s is a compact operator of trace class with trace

$$\operatorname{Tr} s = \int_M \operatorname{Tr} K(x,x).$$

On the other hand, since $d\gamma \in A^2(M)$ is the curvature of ∇, it follows that the connection ∇ on E is flat if and only if the one-form $\gamma \in A^1(M)$ is closed. Assume now that $\gamma \in A^1(M)$ is a closed one-form (so that ∇ is a flat connection and $d_\gamma^2 = 0$). Then, applying (2.9) we get

$$(2.13) \qquad X_t^*\gamma = d \log \lambda_{X_t} + \gamma.$$

Hence, from (2.10), (2.11) and (2.13) (also, by Theorem 2.1) we obtain the formula

$$(2.14) \qquad 1 - s = d_\gamma h + h d_\gamma.$$

Then we have the following result.

THEOREM 2.15: *Let M be a compact manifold, and let $\gamma \in A^1(M)$ be a one-form. Consider in $A(M)$ the continuous operator $d_\gamma = d + \gamma\wedge$. Then there exist two continuous linear maps $s, h\colon A(M) \to A(M)$ of degrees 0 and -1 respectively, such that*

(i) *s is a compact operator;*
(ii) *if γ is a closed one-form, then $1 - s = d_\gamma h + h d_\gamma$.*

Remarks: (1) If $\gamma \in A^1(M)$ is closed, then $\sum_r (-1)^r \operatorname{Tr} s^r$ is the Euler characteristic of the (finite dimensional) cohomology $H_\gamma(M)$ of the complex $(A(M), d_\gamma)$, and h is a parametrix for $(A(M), d_\gamma)$.

(2) Theorem 2.15 generalizes Lemmas 7 and 8 of [27]. That is the case where $\gamma = 0$ (so that $d_\gamma = d$ and $\lambda_{X_t} = 1$).

3. Compact operators for transitive foliations

In this section we discuss the case where M is equipped with a transitive foliation.

Let M be a smooth manifold, and let $\mathcal{F}$ be a smooth foliation on M. Denote by $T\mathcal{F} \subset TM$ the integrable subbundle of vectors of M tangent to $\mathcal{F}$, and by $\mathfrak{X}(\mathcal{F}) = \Gamma T\mathcal{F} \subset \mathfrak{X}(M)$ the Lie subalgebra of vector fields tangent to $\mathcal{F}$. Consider a smooth vector bundle E over M. Then a decreasing filtration $F^k A(M,E)$ by $A(M)$-modules of $A(M,E)$ is given by

(3.1) $$F^k A^r(M,E) = \{\alpha \in A^r(M,E) \mid i(X_1 \wedge \cdots \wedge X_{r-k+1})\alpha = 0, X_i \in \mathfrak{X}(\mathcal{F})\}.$$

Clearly, $F^k A^r(M,E) = F^k A^r(M) \otimes_{A^0(M)} \Gamma E$, where $F^k A(M)$ is the usual decreasing filtration of $A(M) = A(M, M \times \mathbb{R})$. In particular, $F^0 A^r(M,E) = A^r(M,E)$ and $F^{r+1} A^r(M,E) = 0$. This filtration is invariant under the interior products $i(X)$ for $X \in \mathfrak{X}(\mathcal{F})$. Now, let ∇ be a connection on E. Then the filtration is invariant under d_∇ and $\theta_\nabla(X)$ for $X \in \mathfrak{X}(\mathcal{F})$. Hence, if ∇ is flat, then $(A(M,E), d_\nabla)$ together with this filtration is a filtered complex of $A(M)$-modules, so that we have a spectral sequence $(E_i(\nabla), (d_\nabla)_i)$, which converges to $H(M,E)$ after a finite number of steps.

On the other hand, consider the Lie algebra $\mathfrak{X}(M,\mathcal{F}) \subset \mathfrak{X}(M)$ of infinitesimal transformations of $(M,\mathcal{F})$. The foliation $\mathcal{F}$ is called **transitive** if $\mathfrak{X}(M,\mathcal{F}) \subset \mathfrak{X}(M)$ is a transitive space. If M is compact and $\mathcal{F}$ transitive, then it is clear that we can always extract a finite dimensional transitive subspace out of $\mathfrak{X}(M,\mathcal{F})$.

THEOREM 3.2: *Let $\mathcal{F}$ be a transitive foliation on a compact manifold M, and let ∇ be a connection on a smooth vector bundle E over M. Then there exist two continuous linear maps s, h: $A(M,E) \to A(M,E)$ of degrees 0 and -1 respectively, such that*

(i) *s is a compact operator;*
(ii) *$s(F^k A(M,E)) \subset F^k A(M,E)$ for all k;*
(iii) *$h(F^k A(M,E)) \subset F^{k-1} A(M,E)$ for all k;*
(iv) *if ∇ is flat, then $1 - s = d_\nabla h + h d_\nabla$.*

Proof: Consider a finite dimensional transitive space $V \subset \mathfrak{X}(M,\mathcal{F})$, and let g and f be as in Theorem 2.1. Now, we define the operators $s, h\colon A(M,E) \to A(M,E)$ by (2.2). Then, from Theorem 2.1, (i) and (iv) follow.

On the other hand, for each $t \in \mathbb{R}$ and $X \in V$, let $j_t\colon M \to \mathbb{R}\times M$ and $j_X\colon M \to M \times V$ be the inclusion maps. Then, for $Y \in \mathfrak{X}(M)$, $\alpha \in A(\mathbb{R} \times M, \mathbb{R} \times E)$ and $\beta \in A(M \times V, E \times V)$, we obtain

$$(3.3)\qquad \begin{aligned} i(Y)\int_0^1 \tilde{j}_t^{\#}\alpha \cdot dt &= \int_0^1 i(Y)\tilde{j}_t^{\#}\alpha \cdot dt,\\ i(Y)\int_V \tilde{j}_X^{\#}\beta \cdot f(X) \cdot |g| &= \int_V i(Y)\tilde{j}_X^{\#}\beta \cdot f(X) \cdot |g|\,. \end{aligned}$$

A similar result holds for $\theta_\nabla(Y)$. Clearly, for $t \in \mathbb{R}$ and $X, Y \in \mathfrak{X}(M)$, we have

$$(3.4)\qquad i(Y) \circ X_t^{\#} = X_t^{\#} \circ i((X_t)_*Y).$$

Then, since $(X_t)_*Y \in \mathfrak{X}(\mathcal{F})$ for $t \in \mathbb{R}$, $X \in \mathfrak{X}(M,\mathcal{F})$ and $Y \in \mathfrak{X}(\mathcal{F})$, by (3.3) and (3.4), (ii) and (iii) follow. ∎

For $E = M \times \mathbb{R}$, we have $A(M,E) = A(M)$ and $F^kA(M,E) = F^kA(M)$. Then, appplying Theorems 2.15 and 3.2 we obtain the following result.

THEOREM 3.5: *Let $\mathcal{F}$ be a transitive foliation on a compact manifold M, and let $\gamma \in A^1(M)$ be a one-form. Then there exist two continuous linear maps $s, h\colon A(M) \to A(M)$ of degrees 0 and -1 respectively, such that*

(i) *s is a compact operator;*
(ii) *$s(F^kA(M)) \subset F^kA(M)$ for all k;*
(iii) *$h(F^kA(M)) \subset F^{k-1}A(M)$ for all k;*
(iv) *if γ is closed, then $1 - s = d_\gamma h + hd_\gamma$, where $d_\gamma = d + \gamma\wedge\colon A(M) \to A(M)$ is given by $\alpha \mapsto d\alpha + \gamma \wedge \alpha$.*

4. Spectral sequences associated to foliations

In this section we study the spectral sequence $(E_i(\nabla), (d_\nabla)_i)$ of the filtered complex $A(M,E)$ considered in Section 3.

Let M be a smooth manifold of dimension n, and let $\mathcal{F}$ be a smooth foliation of dimension p and codimension q on M. Recall that the spectral sequence $(E_i, d_i) = (E_i(\mathcal{F}), d_i)$ associated to $\mathcal{F}$ arises from the decreasing filtration $F^kA(M)$ by differential ideals of the de Rham complex $(A(M), d)$ of M. Since $F^{q+1}A(M) = 0$ and $F^0A(M) = A(M)$, (E_i, d_i) collapses at the $(q+1)$-th term and converges to the real cohomology $H(M)$ of M. Each E_i has the induced topology being d_i continuous, and obtaining that E_1 in general is not Hausdorff (see [14]).

Now, consider a Riemannian metric on M and the orthogonal complement $Q = T\mathcal{F}^\perp \subset TM$ of $T\mathcal{F}$. Then we obtain the associated bigrading of $A(M)$ given by

$$A^{u,v}(M) = \Gamma(\Lambda^u Q^* \otimes \Lambda^v T^*\mathcal{F}) = \Gamma\Lambda^u Q^* \otimes_{A^0(M)} \Gamma\Lambda^v T^*\mathcal{F}. \tag{4.1}$$

The filtration of $A(M)$ may be represented by $F^k A(M) = \bigoplus_{u\geq k} A^{u,\cdot}(M)$, and the exterior derivative d decomposes as the sum of the homogeneous operators $d_\mathcal{F}$, $d_{1,0}$ and $d_{2,-1}$ of bidegrees $(0,1)$, $(1,0)$ and $(2,-1)$ respectively, which satisfy the usual identities. In particular, $d_\mathcal{F}^2 = 0$. So we obtain the following topological identities of bigraded topological differential algebras:

$$(E_0, d_0) = (A(M), d_\mathcal{F}), \quad (E_1, d_1) = (H(A(M), d_\mathcal{F}), d_{1,0*}). \tag{4.2}$$

It follows that $E_2 \cong H(H(A(M), d_\mathcal{F}), d_{1,0*})$, $E_1^{\cdot,0} = A_b(M)$, and $E_2^{\cdot,0} = H_b(M)$, where $A_b(M) = A^{\cdot,0}(M) \cap \operatorname{Ker} d_\mathcal{F}$ and $H_b(M) = H(A_b(M), d)$ are respectively the differential algebra of basic forms and the basic cohomology of $\mathcal{F}$. $E_1^{0,\cdot} = H(\Gamma\Lambda T^*\mathcal{F}, d_\mathcal{F})$ is the foliated cohomology of $\mathcal{F}$, and $E_1^{\cdot,p}$ and $E_2^{\cdot,p}$ are isomorphic to the transverse complex and the transverse cohomology respectively (cf. [14]). Moreover, $E_2^{\cdot,p}$ is also isomorphic to the $\mathcal{F}$-relative de Rham cohomology (see [26]).

On the other hand, let E be a smooth vector bundle over M. Then we have the associated bigrading of the $A(M)$-module $A(M, E)$ given by

$$A^{u,v}(M, E) = \Gamma(\Lambda^u Q^* \otimes \Lambda^v T^*\mathcal{F} \otimes E) = A^{u,v}(M) \otimes_{A^0(M)} \Gamma E, \tag{4.3}$$

and the filtration $F^k A(M, E)$ of $A(M, E)$ may be represented by

$$F^k A(M, E) = \bigoplus_{u\geq k} A^{u,\cdot}(M, E) = F^k A(M) \otimes_{A^0(M)} \Gamma E. \tag{4.4}$$

Consider now a connection $\nabla\colon \Gamma E = A^0(M, E) \to A^1(M, E)$ on E. Then ∇ decomposes as the sum of the partial connections $\nabla^\mathcal{F}\colon \Gamma E \to A^{0,1}(M, E)$ and $\nabla^{1,0}\colon \Gamma E \to A^{1,0}(M, E)$ on E. It follows that the covariant exterior derivative d_∇ may be decomposed as the sum of the homogeneous operators $d_{\nabla^\mathcal{F}}$, $d_{\nabla^{1,0}}$ and $d_{2,-1} = d_{2,-1} \otimes 1$ of bidegrees $(0,1)$, $(1,0)$ and $(2,-1)$ respectively, where $d_{\nabla^\mathcal{F}}$ (resp. $d_{\nabla^{1,0}}$) is induced by $d_\mathcal{F}$ and $\nabla^\mathcal{F}$ (resp. by $d_{1,0}$ and $\nabla^{1,0}$).

Assume that the vector bundle E is $\mathcal{F}$-foliated with flat partial connection $\nabla^\mathcal{F}$. Then $d_{\nabla^\mathcal{F}}^2 = 0$, so that $d_\nabla^2(F^k A(M, E)) \subset F^{k+1} A(M, E)$ for all k. Therefore, we have the bigraded topological complexes $(E_0(\nabla),(d_\nabla)_0)$ and $(A(M,E),d_{\nabla^{\mathcal{F}}})$, and the bigraded topological space $E_1(\nabla)$. So we obtain the following result.

PROPOSITION 4.5: *We have the following topological identities,*

$$(4.6)\qquad (E_0(\nabla),(d_\nabla)_0)=(A(M,E),d_{\nabla^{\mathcal{F}}}),\ \ E_1(\nabla)=H(A(M,E),d_{\nabla^{\mathcal{F}}}),$$

of bigraded topological complexes and bigraded topological spaces respectively.

We define the graded $A_b(M)$-module $A_b(M,E)$ of E-valued basic forms of $\mathcal{F}$ by

$$\begin{aligned}(4.7)\ A_b(M,E) &= E_1^{\cdot,0}(\nabla)=A^{\cdot,0}(M,E)\cap \operatorname{Ker} d_{\nabla^{\mathcal{F}}}\\ &= \{\alpha\in A(M,E)\mid i(X)\alpha=\theta_\nabla(X)\alpha=0 \quad \text{for } X\in\mathfrak{X}(\mathcal{F})\}.\end{aligned}$$

In particular, $A_b^0(M,E)\subset \Gamma E$ is the $A_b^0(M)$-module of $\mathcal{F}$-foliated sections of E. $E_1^{0,\cdot}(\nabla)=H(\Gamma L(\Lambda T\mathcal{F},E),d_{\nabla^{\mathcal{F}}})$ is the E-valued foliated cohomology of $\mathcal{F}$.

Now, denote by $\mathcal{A}(M)$, $\mathcal{A}_b(M)$, $\mathcal{A}(M,E)$ and $\mathcal{A}_b(M,E)$ the corresponding sheaves of germs. Then, for each u, $0\le u\le q$,

$$(\mathcal{A}^{u,\cdot}(M,E),d_{\nabla^{\mathcal{F}}})\cong(\mathcal{A}^{u,\cdot}(M)\otimes_{\mathcal{A}_b^0(M)}\mathcal{A}_b^0(M,E),d_{\mathcal{F}}\otimes 1)$$

is a fine resolution of the sheaf $\mathcal{A}_b^u(M,E)\cong\mathcal{A}_b^u(M)\otimes_{\mathcal{A}_b^0(M)}\mathcal{A}_b^0(M,E)$ (cf. [32]). Thus

$$(4.8)\qquad E_1^{u,\cdot}(\nabla)=H(M,\mathcal{A}_b^u(M,E)).$$

It follows that $E_1^{0,\cdot}(\nabla)=H(M,\mathcal{A}_b^0(M,E))$.

For example, the normal bundle $E=\nu\mathcal{F}=TM/T\mathcal{F}$ of $\mathcal{F}$ is canonically $\mathcal{F}$-foliated by the partial Bott connection $\nabla^{\mathcal{F}}$. The elements of $A_b^0(M,\nu\mathcal{F})$ are the transverse fields associated to the infinitesimal transformations of $\mathcal{F}$, and the elements of $E_1^{0,1}(\nabla)=H^1(M,\mathcal{A}_b^0(M,\nu\mathcal{F}))$ may be interpreted as infinitesimal deformations of $\mathcal{F}$ (see [15]).

For $E=\Lambda^u\nu^*\mathcal{F}$ and $\nabla_X^{\mathcal{F}}=\theta(X)$, $X\in\mathfrak{X}(\mathcal{F})$, the partial Bott connection, we have $A_b^u(M)=A_b^0(M,\Lambda^u\nu^*\mathcal{F})$, $(E_0^{u,\cdot},d_0)=(E_0^{0,\cdot}(\nabla),(d_\nabla)_0)$, and $E_1^{u,\cdot}=E_1^{0,\cdot}(\nabla)=H(M,\mathcal{A}_b^u(M))$. In particular, $E_1^{0,\cdot}=H(M,\mathcal{A}_b^0(M))$. Since $(\mathcal{A}_b(M),d)$ is a resolution of the constant real sheaf $\mathcal{R}$, it follows that $E_2^{u,\cdot}=H^u(H(M,\mathcal{A}_b(M)))$.

Suppose now that E is a flat vector bundle over M with flat connection ∇. The spectral sequence $(E_i(\nabla),(d_\nabla)_i)$ associated to $\mathcal{F}$ and E arises from the decreasing filtration $F^kA(M,E)$ by $A(M)$-modules of the complex $(A(M,E),d_\nabla)$. $(E_i(\nabla),(d_\nabla)_i)$ collapses at the $(q+1)$-th term and converges to the cohomology

$H(M,E)$. It is clear that the multiplication map $E_i \otimes E_i(\nabla) \to E_i(\nabla)$ is a homomorphism of complexes, and that $E_i(\nabla)$ has the induced topology being $(d_\nabla)_i$ continuous for all i. $E_1(\nabla)$ in general is not Hausdorff.

Since $d_\nabla^2 = 0$, the homogeneous operators $d_{\nabla^{\mathcal{F}}}$, $d_{\nabla^{1,0}}$ and $d_{2,-1}$ satisfy

$$\begin{aligned} &d^2_{\nabla^{\mathcal{F}}} = d^2_{2,-1} = d_{\nabla^{\mathcal{F}}} d_{\nabla^{1,0}} + d_{\nabla^{1,0}} d_{\nabla^{\mathcal{F}}} = 0, \\ &d_{\nabla^{1,0}} d_{2,-1} + d_{2,-1} d_{\nabla^{1,0}} = d^2_{\nabla^{1,0}} + d_{2,-1} d_{\nabla^{\mathcal{F}}} + d_{\nabla^{\mathcal{F}}} d_{2,-1} = 0. \end{aligned} \tag{4.9}$$

Hence we obtain the following result.

PROPOSITION 4.10: *We have the identities* (4.6) *and the following topological identity of bigraded topological complexes:*

$$(E_1(\nabla),(d_\nabla)_1) = (H(A(M,E),d_{\nabla^{\mathcal{F}}}),d_{\nabla^{1,0}*}). \tag{4.11}$$

Therefore, $E_2(\nabla) \cong H(H(A(M,E),d_{\nabla^{\mathcal{F}}}),d_{\nabla^{1,0}*})$, $(E_1^{\cdot,0}(\nabla),(d_\nabla)_1) = (A_b(M,E),d_\nabla)$, and $E_2^{\cdot,0}(\nabla) = H_b(M,E)$ is the E-valued basic cohomology of $\mathcal{F}$. In particular, $H_b^0(M,E) = H^0(M,E)$ is the vector space of parallel (or locally constant) sections of E. Clearly, the canonical map $H_b^k(M,E) \to H^k(M,E)$ is injective for $k=1$. Similarly, $E_2^{\cdot,p}(\nabla)$ is isomorphic to the E-valued $\mathcal{F}$-relative de Rham cohomology.

Let $\mathcal{E}$ be the sheaf of germs of parallel sections of E. Then $(\mathcal{A}(M,E),d_\nabla) \cong (\mathcal{A}(M)\otimes\mathcal{E}, d\otimes 1)$ is a fine resolution of the sheaf $\mathcal{E}$. It follows that $H(M,E) = H(M,\mathcal{E})$.

PROPOSITION 4.12: *For each $u \geq 0$ we have:*

$$E_1^{u,\cdot}(\nabla) = H(M,\mathcal{A}_b^u(M)\otimes\mathcal{E}), \quad E_2^{u,\cdot}(\nabla) = H^u(H(M,\mathcal{A}_b(M)\otimes\mathcal{E})). \tag{4.13}$$

Proof: It is easy to check that

$$(\mathcal{A}^{u,\cdot}(M,E),d_{\nabla^{\mathcal{F}}}) \cong (\mathcal{A}^{u,\cdot}(M)\otimes\mathcal{E}, d_{\mathcal{F}}\otimes 1)$$

is a fine resolution of the sheaf $\mathcal{A}_b^u(M,E) \cong \mathcal{A}_b^u(M)\otimes\mathcal{E}$. It follows that $E_1^{u,\cdot}(\nabla) = H(M,\mathcal{A}_b^u(M)\otimes\mathcal{E})$.

Clearly, $(\mathcal{A}_b(M,E),d_\nabla) \cong (\mathcal{A}_b(M)\otimes\mathcal{E}, d\otimes 1)$ is a resolution of the sheaf $\mathcal{E}$, and $(E_1(\nabla),(d_\nabla)_1) = (H(M,\mathcal{A}_b(M)\otimes\mathcal{E}),(d\otimes 1)_*)$. The desired result follows. ∎

Now, let $E = M\times\mathbb{R}$ be the trivial vector bundle over M, and consider two one-forms $\gamma,\gamma' \in A^1(M)$. Let ∇ (resp. ∇') be the connection on E with connection form γ (resp. γ') with respect to the smooth section $\sigma \in \Gamma E$ defined by $\sigma(x) =$

$(x,1)$. Then it is easy to see that $\gamma - \gamma' \in d(A^0(M))$ if and only if there exists an automorphism of vector bundles $\tilde{\varphi}: E \overset{\cong}{\to} E$ inducing the identity map in M such that $\tilde{\varphi}^{\#} \circ \nabla = \nabla' \circ \tilde{\varphi}^{\#}$.

Therefore, if $\gamma \in A^1(M)$ is a closed one-form, then each $H_\gamma(M)$ and $E_i(\gamma)$ depends only on the class $[\gamma] \in H^1(M)$. In particular, if $\gamma \in A_b^1(M)$ is a closed basic one-form, then each $H_\gamma(M)$ and $E_i(\gamma)$ depends only on the class $[\gamma] \in H_b^1(M) \subset H^1(M)$, and we have

$$(4.14) \qquad (E_0(\gamma), (d_\gamma)_0) = (E_0, d_0), \qquad (E_1(\gamma), (d_\gamma)_1) = (E_1, d_1 + \gamma\wedge).$$

The following proposition is easily verified.

PROPOSITION 4.15: *Let $\gamma \in A^1(M)$ be a closed one-form, and let ∇ be the flat connection on $E = M \times \mathbb{R}$ with connection form γ with respect to σ. Assume that M has a finite number of connected components. Then the following conditions are equivalent:*

(i) *(E, ∇) is trivial as a flat vector bundle;*
(ii) *the class $[\gamma] = 0 \in H^1(M)$;*
(iii) *$E_2^{0,0}(\gamma) \cong E_2^{0,0}$;*
(iv) *$H_\gamma(M) \cong H(M)$;*
(v) *$E_i(\gamma) \cong E_i$ for all i.*

Remark: Let $\gamma \in A^1(M)$ be a closed one-form. Then it is easy to check that $E_2^{0,0}(\gamma) \cong E_2^{0,0}(-\gamma)$. Assume now that M is connected. Then we have $E_2^{0,0}(\pm\gamma) \cong E_2^{0,0} = \mathbb{R}$ or $E_2^{0,0}(\pm\gamma) = 0$. The first case occurs if and only if the class $[\gamma] = 0 \in H^1(M)$.

5. Finiteness theorem for transitive foliations

In this section we prove a finiteness theorem for the spectral sequence of the filtered complex $A(M, E)$ considered in Section 4. For this purpose, we shall use the Riesz theory of compact operators [13, 25].

First, we consider the following more general case. Let (A, d) be a filtered (cochain) complex of Hausdorff locally convex topological vector spaces A^r over the real field (or the complex field) and continuous linear maps d^r with decreasing filtration $F^kA = \bigoplus_r F^kA^r$ (stable under d) such that $A = \bigoplus_{r=0}^n A^r$ (with n integer), $F^kA \subset A$ are closed subspaces, $F^0A^r = A^r$ and $F^kA^r = 0$ for $k > r$, where on A we consider the direct sum topology. Then, we have a spectral sequence (E_i, d_i) which converges to the cohomology $H(A)$ of (A, d) after a finite number of steps. Each E_i, E_∞ and $H(A)$ with the induced topology is a locally convex topological vector space and d_i is continuous. Moreover, E_0 is Hausdorff and the canonical isomorphism $E_1 \to H(E_0) = H(E_0, d_0)$ (resp. $E_{i+1} \to H(E_i) = H(E_i, d_i)$, $i \geq 1$) is a topological isomorphism (resp. is continuous). E_1 in general is not Hausdorff obtaining two new topological complexes, the closure $\bar{O}$ of the trivial subspace of E_1, and $\mathbb{E}_1 = E_1/\bar{O}$, so that $\mathbb{E}_1$ is Hausdorff and we have the exact sequence of topological complexes $0 \to \bar{O} \to E_1 \xrightarrow{\tau} \mathbb{E}_1 \to 0$. We will say that $\mathbb{E}_1$ is the **reduction** of E_1, and let $\mathbb{E}_2 = H(\mathbb{E}_1)$ be its cohomology.

Definition 5.1: A pair of continuous linear maps s, h: $A \to A$ of degrees 0 and -1 respectively will be called a **2-parametrix** for A if

(i) s is a compact operator;
(ii) $s(F^kA) \subset F^kA$ for all k;
(iii) $h(F^kA) \subset F^{k-1}A$ for all k;
(iv) $1 - s = dh + hd$.

LEMMA 5.2: *Assume that there exists a 2-parametrix s, h for A. Then we have:*

(i) *There is a finite dimensional topological filtered subcomplex $K \subset A$ with spectral sequence $(E_i(K), d_i)$ such that the induced linear maps $E_2(K) \to E_2$ and $E_2(K) \to \mathbb{E}_2$ are topological isomorphisms.*
(ii) *E_2 and $\mathbb{E}_2$ are finite dimensional Hausdorff, and the canonical map τ_*: $E_2 \to \mathbb{E}_2$ is a topological isomorphism, so that $H(\bar{O}) = 0$.*
(iii) *Each $E_i \cong E_i(K)$, $2 \leq i \leq \infty$, and $H(A) \cong H(K)$ is finite dimensional and the induced topology coincides with the Euclidean topology. In particular, the identities $E_{i+1} \equiv H(E_i)$, $0 \leq i < \infty$, are also topological.*

Proof: Since $F^kA \subset A$ is a closed subspace, s: $A \to A$ defines a compact operator s: $F^kA \to F^kA$ for each $k = 0, \dots, n$. From [13, 25] it follows that $1 - s$: $F^kA \to F^kA$ has finite ascent and finite descent m_k for all $k = 0. \dots, n$. Let m be the maximum of the m_k, $0 \leq k \leq n$. Then the kernel $K = \mathrm{Ker}(1-s)^m$ and the image $I = (1-s)^m(A)$ of $(1-s)^m$: $A = F^0A \to A = F^0A$ are topological filtered subcomplexes of A with filtrations $F^kK = \mathrm{Ker}\left((1-s)^m|_{F^kA}\right) = K \cap F^kA$ and $F^kI = (1-s)^m(F^kA)$. Furthermore, $A = K \oplus I$ as a topological filtered complex with $F^kA = F^kK \oplus F^kI$ (as a topological complex), K is finite dimensional, F^kK and F^kI are stable under s, $(1-s)^m = 0$: $F^kK \to F^kK$, and $1 - s$: $F^kI \to F^kI$ is a topological isomorphism for each $k = 0, \dots, n$.

Now, let $E_i(K)$ and $E_i(I)$ be the spectral sequences of K and I respectively. It is clear that $E_i(K)$, $0 \leq i \leq \infty$, and $H(K)$ are finite dimensional Hausdorff. Then

$$E_1 \cong E_1(K) \oplus E_1(I), \qquad \mathbb{E}_1 \cong E_1(K) \oplus \mathbb{E}_1(I)$$

as topological complexes, where $\mathbb{E}_1(I) = E_1(I)/\bar{O}$. Consider the canonical projection $\pi_I = \left((1-s)^m|_I\right)^{-1} \circ (1-s)^m \colon A \to I$. So we obtain a 2-parametrix $s|_I, \pi_I \circ (h|_I) \colon I \to I$ for I. Since $1 - s|_I \colon I \to I$ is a topological isomorphism, we have $E_2(I) = \mathbb{E}_2(I) = 0$, where $\mathbb{E}_2(I) = H(\mathbb{E}_1(I))$. This implies that

$$E_2(K) \cong E_2(K) \oplus E_2(I) \cong E_2, \quad E_2(K) \cong E_2(K) \oplus \mathbb{E}_2(I) \cong \mathbb{E}_2$$

as topological complexes. The desired result follows. ∎

THEOREM 5.3: *Let $\mathcal{F}$ be a transitive foliation on a compact manifold M, and let ∇ be a flat connection on a smooth vector bundle E over M. Then the spectral sequence $(E_i(\nabla), (d_\nabla)_i)$ of the filtered complex $(A(M,E), d_\nabla)$ satisfies:*

(i) *There exists a finite dimensional topological filtered subcomplex $K \subset A(M,E)$ with spectral sequence $E_i(K)$ such that the induced linear maps $E_2(K) \to E_2(\nabla)$ and $E_2(K) \to \mathbb{E}_2(\nabla)$ are topological isomorphisms, where $\mathbb{E}_2(\nabla) = H(\mathbb{E}_1(\nabla))$ and $\mathbb{E}_1(\nabla) = E_1(\nabla)/\bar{O}$.*

(ii) *$E_2(\nabla)$ and $\mathbb{E}_2(\nabla)$ are finite dimensional Hausdorff, and $E_2(\nabla) \cong \mathbb{E}_2(\nabla)$ canonically and topologically, so that $H(\bar{O}) = 0$.*

(iii) *Each $E_i(\nabla) \cong E_i(K)$, $2 \leq i \leq \infty$, and $H(M,E) \cong H(K)$ is finite dimensional and its topology is the Euclidean topology. The identities $E_{i+1}(\nabla) \equiv H(E_i(\nabla))$, $0 \leq i < \infty$, are also topological.*

Proof: Consider the 2-parametrix s, h for $A(M,E)$ constructed in Theorem 3.2 and apply Lemma 5.2. ∎

From Theorem 5.3 (also, by Theorem 3.5 and Lemma 5.2) we obtain the following result.

THEOREM 5.4: *Let $\mathcal{F}$ be a transitive foliation on a compact manifold M, and let $\gamma \in A^1(M)$ be a closed one-form. Then the spectral sequence $(E_i(\gamma), (d_\gamma)_i)$ of the filtered complex $(A(M), d_\gamma) = (A(M, M \times \mathbb{R}), d_\nabla)$ satisfies the properties* (i), (ii) *and* (iii) *of Theorem* 5.3, *where $d_\gamma = d + \gamma\wedge$.*

Remark: For $\gamma = 0$, the results obtained above are reduced to the ordinary case of [27] and [19, Section 1].

From Theorems 5.3 and 5.4 we have for $T\mathcal{F} = TM$ (also, for $T\mathcal{F} = 0$) the following result.

COROLLARY 5.5: *Let E be a flat vector bundle over a compact manifold M. Then there exists a finite dimensional topological subcomplex $K \subset A(M,E)$ such that $H(K) \cong H(M,E)$ topologically. In particular, for each closed one-form $\gamma \in A^1(M)$, there is a finite dimensional topological subcomplex (K, d_γ) of*

$(A(M), d_\gamma)$ such that $H(K, d_\gamma) \cong H_\gamma(M)$ topologically. Clearly, $H(M, E)$ and $H_\gamma(M)$ are finite dimensional Hausdorff.

6. Examples

In this section we compute some examples of homogeneous Lie foliations on compact connected homogeneous spaces.

Let $\mathcal{F}$ be a smooth foliation of codimension q on a smooth manifold M, and let $\pi\colon TM \to \nu\mathcal{F} = TM/T\mathcal{F}$ be the canonical projection. Then each $X \in \mathfrak{X}(M)$ determines a smooth section $\bar{X} = \pi(X) \in \Gamma\nu\mathcal{F}$. On says that $\mathcal{F}$ is **transversally parallelizable** if there exist elements $X_1, \dots, X_q \in \mathfrak{X}(M, \mathcal{F})$ such that $\bar{X}_1, \dots, \bar{X}_q \in \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F}) = A^0_b(M, \nu\mathcal{F})$ are linearly independent at each point of M. The set $\mathcal{P} = \{\bar{X}_1, \dots, \bar{X}_q\}$ is called a **transverse parallelism** of $\mathcal{F}$. If the q-dimensional vector space generated by $\mathcal{P}$ is a Lie subalgebra $\mathfrak{g}$ of the Lie algebra $\mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$, then $\mathcal{F}$ is called a **Lie $\mathfrak{g}$-foliation**, and $\mathcal{P}$ is called a **transverse Lie parallelism** of $\mathcal{F}$. It is clear that every transversally parallelizable foliation is transitive. Similarly, every foliation defined by the fibers of a locally trivial fibration is transitive. Note also that the canonical lift of a Riemannian foliation to the bundle of its orthonormal transverse frames is a transversally parallelizable foliation (see [21, 22]).

Now, let $\mathcal{F}$ be a transversally parallelizable foliation of codimension q on M, and consider a transverse parallelism $\mathcal{P} = \{\bar{X}_1, \dots, \bar{X}_q\}$ of $\mathcal{F}$. Then $\mathcal{P}$ determines an $\mathcal{F}$-basic connection $\nabla = \nabla^{\mathcal{P}}$ on $\nu\mathcal{F}$ given by

$$\nabla_X \bar{Z} = \pi[X_{\mathcal{F}}, Z] + \sum_{i=1}^{q} f_i \pi[X_i, Z] = \sum_{i=1}^{q} Z(f_i)\bar{X}_i + \pi[X, Z] \tag{6.1}$$

for $X, Z \in \mathfrak{X}(M)$, where $X_i \in \mathfrak{X}(M, \mathcal{F})$ represents $\bar{X}_i$, and $X_{\mathcal{F}} \in \mathfrak{X}(\mathcal{F})$ and $f_i \in A^0(M)$, $i = 1, \dots, q$, are given by $X = X_{\mathcal{F}} + \sum_{i=1}^{q} f_i X_i$. It is easy to see that ∇ depends only on the q-dimensional vector space V generated by $\mathcal{P}$, and that ∇ is flat if $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation, where $\mathfrak{g} = V$. Conversely, if M is connected and ∇ is flat, then $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation. Moreover, if M is connected, then $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation with dense leaves if and only if ∇ is independent of the choice of $\mathcal{P}$. In this case the canonical flat connection ∇ is completely characterized by the formula

$$\nabla_X \bar{Z} = \pi[X, Z] \quad \text{for } X \in \mathfrak{X}(M, \mathcal{F}), \quad Z \in \mathfrak{X}(M). \tag{6.2}$$

Next, let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation on M, and consider a transverse Lie parallelism $\mathcal{P} = \{\bar{X}_1, \dots, \bar{X}_q\}$ of $\mathcal{F}$, where $X_i \in \mathfrak{X}(M, \mathcal{F})$ represents $\bar{X}_i$, $i = 1, \dots, q$. Suppose that the q_0-dimensional vector space generated by $\bar{X}_1, \dots, \bar{X}_{q_0}$, $0 \le q_0 \le q$, is an ideal $\mathfrak{h}$ of $\mathfrak{g}$. Then $T\mathcal{F}$ and $X_1, \dots, X_{q_0}$ define a Lie $\mathfrak{g}/\mathfrak{h}$-foliation $\mathcal{F}_\mathfrak{h}$ of codimension $q - q_0$ on M with $\mathcal{F} \subset \mathcal{F}_\mathfrak{h}$, and the flat connection $\nabla = \nabla^\mathfrak{g}$ on $\nu\mathcal{F}$ of $\mathcal{F}$ induces a flat connection $\nabla^\mathfrak{h}$ on the $\mathcal{F}$-foliated normal bundle

$$(6.3) \qquad Q_\mathfrak{h} = T\mathcal{F}_\mathfrak{h}/T\mathcal{F} \subset \nu\mathcal{F} \quad \text{of } \mathcal{F} \quad \text{in } \mathcal{F}_\mathfrak{h}$$

given by

$$(6.4) \qquad \nabla^\mathfrak{h}_X \bar{Z} = \pi_\mathfrak{h}[X_\mathcal{F}, Z] + \sum_{i=1}^{q} f_i \pi_\mathfrak{h}[X_i, Z]$$

for $X \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(\mathcal{F}_\mathfrak{h})$, where $\pi_\mathfrak{h}$: $T\mathcal{F}_\mathfrak{h} \to Q_\mathfrak{h}$ is the canonical projection, and $X_\mathcal{F}$ and f_i are given as above. Clearly, the flat connection $\nabla^{\mathfrak{g}/\mathfrak{h}}$ on $\nu\mathcal{F}_\mathfrak{h}$ of $\mathcal{F}_\mathfrak{h}$ is also induced by ∇, and the canonical projection $\nu\mathcal{F} \to \nu\mathcal{F}_\mathfrak{h}$ is compatible with ∇ and $\nabla^{\mathfrak{g}/\mathfrak{h}}$.

On the other hand, let $\mathcal{F}$ be a transitive foliation on a compact connected manifold M. Then P. Molino has proved in [20, 22] that the closures of the leaves of $\mathcal{F}$ are the fibers of a locally trivial fibration π_b: $M \to W$, called the **basic fibration**, and the restriction of $\mathcal{F}$ to each fiber of π_b is a Lie $\mathfrak{g}$-foliation with dense leaves, where $\mathfrak{g}$ is an algebraic invariante of $\mathcal{F}$, called the **structural Lie algebra**. Consider now the basic foliation $\mathcal{F}_b$ defined by π_b, whose leaves are the closures of the leaves of $\mathcal{F}$, so that $\mathcal{F} \subset \mathcal{F}_b$. Then an easy computation shows that the $\mathcal{F}$-foliated normal bundle

$$(6.5) \qquad \mathcal{C}(\mathcal{F}) = T\mathcal{F}_b/T\mathcal{F} \subset \nu\mathcal{F} \quad \text{of } \mathcal{F} \quad \text{in } \mathcal{F}_b$$

of dimension $q_0 = \dim \mathfrak{g}$ is flat, whose canonical flat connection $\nabla = \nabla^\mathcal{C}$ is completely characterized by the formula

$$(6.6) \quad \nabla_X \bar{Z} = \pi_0[X, Z] \quad \text{for } X \in \mathfrak{X}(M, \mathcal{F}), Z \in \mathfrak{X}(\mathcal{F}_b), \bar{Z} = \pi_0(Z) \in \Gamma\mathcal{C}(\mathcal{F}),$$

where π_0: $T\mathcal{F}_b \to \mathcal{C}(\mathcal{F})$ is the canonical projection. The Molino commuting sheaf (or central transverse sheaf) of $\mathcal{F}$ (cf. [20, 22]) is the sheaf of germs of parallel sections of the flat vector bundle $\mathcal{C}(\mathcal{F})$, and if $\mathcal{F}$ is transversally parallelizable, then ∇ is induced by the connection on $\nu\mathcal{F}$ given by (6.1).

A method to construct examples of Lie foliations is the following. Let G and G_1 be two simply connected Lie groups, and let D: $G_1 \to G$ be a surjective homomorphism of Lie groups. Suppose that G_1 contains a discrete uniform subgroup Γ_1. Then the foliation $\tilde{\mathcal{F}}$ on G_1 by the fibers of D, which is also defined by Ker D, is invariant by left translations by the elements of Γ_1. Hence, $\tilde{\mathcal{F}}$ induces a foliation $\mathcal{F}$ on the compact connected homogeneous space $M = \Gamma_1 \backslash G_1$ such that

$\tilde{\mathcal{F}} = \pi^*\mathcal{F}$, where $\pi: G_1 \to M$ is the universal covering of M. Clearly, $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation and D is the developing map of $\mathcal{F}$, $h = D|_{\Gamma_1} : \Gamma_1 \to G$ is the holonomy representation and $\Gamma = h(\Gamma_1) \subset G$ is the holonomy group of $\mathcal{F}$, where $\mathfrak{g}$ is the Lie algebra of G. Now, let K be the closure of Γ in G, and consider the homogeneous space $W = K\backslash G$. Then the canonical projection $\pi_b: M = \Gamma_1\backslash G_1 \to W = K\backslash G$ induced by D is the basic fibration of $\mathcal{F}$ and the Lie algebra of the Lie group K is its structural Lie algebra. Such a Lie $\mathfrak{g}$-foliation is called **homogeneous**.

Now, we apply the preceding results to compute the following example of homogeneous Lie foliations.

Example 1: Let A be the matrix in $\mathrm{SL}(4,\mathbb{Z})$ given by

$$A = \left(\begin{array}{c|c} I & 0 \\ \hline I & I \end{array}\right), \qquad \text{so that } A^t = \left(\begin{array}{c|c} I & 0 \\ \hline tI & I \end{array}\right) \qquad \text{for all } t \in \mathbb{R},$$

where I is the 2×2 identity matrix. Let $G_A = \mathbb{R}\times_\phi \mathbb{R}^4$ be the semidirect product of the additive Lie groups $\mathbb{R}$ and $\mathbb{R}^4$ via the representation $\phi: \mathbb{R} \to \mathrm{SL}(4,\mathbb{R})$ defined by $\phi(t) = A^t$; that is, $G_A = (\mathbb{R}^5, \cdot)$ with the group operation given by

$$\begin{aligned}
&(t, x_1, x_2, x_3, x_4) \cdot (t', x_1', x_2', x_3', x_4') \\
&= (t + t', (x_1, x_2, x_3, x_4) + A^t(x_1', x_2', x_3', x_4')) \\
&= (t + t', x_1 + x_1', x_2 + x_2', x_3 + x_3' + tx_1', x_4 + x_4' + tx_2').
\end{aligned}$$

So we have constructed a simply connected nilpotent Lie group G_A of dimension 5, which is not abelian. It is easy to check that $\Gamma_A = (\mathbb{Z}^5, \cdot) \subset G_A$ is a discrete uniform and torsion-free subgroup, and that the canonical projection $\pi: G_A \to \Gamma_A\backslash G_A$ is the universal covering of the compact connected homogeneous space $M = \Gamma_A\backslash G_A$ of dimension 5. Moreover, M is the quotient manifold $\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4$ of $\mathbb{R} \times \mathbb{T}^4$ by the equivalence relation given by $(t, x) \sim (t + 1, A(x))$, $t \in \mathbb{R}$, $x \in \mathbb{T}^4$, where A also denotes the automorphism of $\mathbb{T}^4$ induced by A. Note that $\pi_S: \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4 \to \mathbb{S}^1$ is a flat bundle with fiber $\mathbb{T}^4$, whose monodromy is given by A, where π_S is induced by the canonical projection of $\mathbb{R} \times \mathbb{T}^4$ onto $\mathbb{R}$.

Now, let $\alpha \in \mathbb{R}$ be a real number, and consider the orthogonal basis of the Euclidean space $\mathbb{R}^4 \equiv (\mathbb{R}^4, \langle,\rangle)$ given by

$$v_1 = (-\alpha, 1, 0, 0), \quad v_2 = (0, 0, -\alpha, 1), \quad v_3 = (1, \alpha, 0, 0), \quad v_4 = (0, 0, 1, \alpha),$$

which satisfies the identities

$$A^t(v_1) = v_1 + tv_2, \;\; A^t(v_2) = v_2, \;\; A^t(v_3) = v_3 + tv_4, \;\; A^t(v_4) = v_4 \;\; \text{for all } t \in \mathbb{R}.$$

Then a basis of left invariant vector fields on G_A is given by

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = -\alpha\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + t\left(-\alpha\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right), \quad X_2 = -\alpha\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},$$
$$X_3 = \frac{\partial}{\partial x_1} + \alpha\frac{\partial}{\partial x_2} + t\left(\frac{\partial}{\partial x_3} + \alpha\frac{\partial}{\partial x_4}\right), \quad X_4 = \frac{\partial}{\partial x_3} + \alpha\frac{\partial}{\partial x_4},$$

where X_i is induced by v_i, $i = 1,2,3,4$. For each $i = 0,1,2,3,4$, X_i defines a vector field, also denoted X_i, on $M = \Gamma_A\backslash G_A = \mathbb{R}\times_{\mathbb{Z}}\mathbb{T}^4$, and $X_0, X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$ is a parallelism on M satisfying

$$(6.7) \qquad \begin{array}{ll} [X_0, X_1] = X_2, & [X_0, X_3] = X_4, \\ [X_i, X_j] = 0 & \text{otherwise.} \end{array}$$

The dual basis $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4 \in A^1(M)$ of X_0, X_1, X_2, X_3, X_4 is given by

$$\omega_0 = dt, \quad \omega_1 = c(-\alpha dx_1 + dx_2), \quad \omega_2 = c(-t(-\alpha dx_1 + dx_2) - \alpha dx_3 + dx_4),$$
$$\omega_3 = c(dx_1 + \alpha dx_2), \quad \omega_4 = c(-t(dx_1 + \alpha dx_2) + dx_3 + \alpha dx_4),$$

where $c = 1/(1+\alpha^2)$. Hence we have

$$(6.8) \qquad d\omega_2 = -\omega_0\wedge\omega_1, \quad d\omega_4 = -\omega_0\wedge\omega_3, \quad d\omega_0 = d\omega_1 = d\omega_3 = 0.$$

Similarly, by (6.7) it follows that X_3, X_4 (resp. X_1, X_2) define a homogeneous Lie foliation $\mathcal{F}$ (resp. $\mathcal{F}_1$) of dimension 2 on M. Consider for example $\mathcal{F}$, since the same techniques can be used for $\mathcal{F}_1$. Then X_0, X_1, X_2 define a transverse Lie parallelism of $\mathcal{F}$. Let $G = (\mathbb{R}^3, \cdot)$ be the Heisenberg group of dimension 3, whose group operation is given by

$$(t, x, y)\cdot(t', x', y') = (t + t', x + x', y + y' + tx').$$

Then the surjective homomorphism of Lie groups $D\colon G_A \to G$ given by

$$(6.9) \qquad D(t, x_1, x_2, x_3, x_4) = (t, -\alpha x_1 + x_2, -\alpha x_3 + x_4)$$

is the developing map of $\mathcal{F}$. Therefore, $\mathcal{F}$ is a homogeneous Lie $\mathfrak{g}$-foliation, $\Gamma = D(\Gamma_A) \subset G$ is its holonomy group and the induced map $\pi_b\colon M \to W = K\backslash G$ is its basic fibration, where $\mathfrak{g}$ is the Lie algebra of G, which is defined by $\bar{X}_0, \bar{X}_1, \bar{X}_2$, and $K = \bar{\Gamma} \subset G$ denotes the closure of Γ in G. Clearly, $\bar{X}_1, \bar{X}_2$ generate an ideal $\mathfrak{h}$ of dimension 2 of $\mathfrak{g}$. Then $T\mathcal{F}$ and X_1, X_2 define a Lie $\mathfrak{g}/\mathfrak{h}$-foliation $\mathcal{F}_{\mathfrak{h}}$ of dimension 4 on M with $\mathcal{F} \subset \mathcal{F}_{\mathfrak{h}}$. It follows that $\mathcal{F}_{\mathfrak{h}}$ is defined by X_1, X_2, X_3, X_4, and its leaves are the fibers of $\pi_S\colon M = \mathbb{R}\times_{\mathbb{Z}}\mathbb{T}^4 \to \mathbb{S}^1$. Furthermore, $\bar{X}_1, \bar{X}_2 \in \Gamma Q_{\mathfrak{h}}$ is a basis of sections for $Q_{\mathfrak{h}} = T\mathcal{F}_{\mathfrak{h}}/T\mathcal{F}$ and the connection form (θ_{ij}) of the flat connection $\nabla = \nabla^{\mathfrak{h}}$ on $Q_{\mathfrak{h}}$ with respect to $\bar{X}_1, \bar{X}_2$ is given by

$$\theta_{11} = \theta_{12} = \theta_{22} = 0, \qquad \theta_{21} = \omega_0 \in A^1(\mathbb{S}^1) \subset A^1(M). \tag{6.10}$$

Consider now the spectral sequence (E_i, d_i) (resp. $(E_i(\nabla), (d_\nabla)_i)$) associated to $\mathcal{F}$ (resp. to $\mathcal{F}$ and $Q_{\mathfrak{h}}$). Then by (6.10) we have

$$E_i(\nabla) = E_i \oplus E_i \qquad \text{for } i = 0, 1. \tag{6.11}$$

On the other hand, suppose that $\alpha \in \mathbb{Q}$ is a rational number, so that $\alpha = a_0/a$ with $a_0, a \in \mathbb{Z}$, $a > 0$ and a_0, a relatively prime. Then $\Gamma \subset G$ is closed, $K = \Gamma$, π_b: $M \to \Gamma\backslash G$ is the basic fibration of $\mathcal{F}$, and the leaves of $\mathcal{F}$ are the fibers of π_b, which are diffeomorphic to $\mathbb{T}^2$. Now, to compute π_b, consider the automorphism of Lie groups ϕ_a: $G \overset{\cong}{\to} G$ defined by $\phi_a(t, x, y) = (t, ax, ay)$, which satisfies $\phi_a(\Gamma) = (\mathbb{Z}^3, \cdot) \subset G$. Then we have the induced diffeomorphism $\Gamma\backslash G \to (\mathbb{Z}^3, \cdot)\backslash G = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2$, where $\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2$ is the compact Heisenberg manifold of dimension 3; that is, the quotient manifold of $\mathbb{R} \times \mathbb{T}^2$ by the equivalence relation given by $(t, x, y) \sim (t + 1, x, y + x)$. Evidently, the canonical projection $\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2 \to \mathbb{S}^1$ is a flat bundle with fiber $\mathbb{T}^2$. Denote also by D: $G_A \to G$ the developing map of $\mathcal{F}$ given by the composition of D with ϕ_a, which is defined by

$$D(t, x_1, x_2, x_3, x_4) = (t, -a_0 x_1 + a x_2, -a_0 x_3 + a x_4). \tag{6.12}$$

It follows that the induced map

$$\pi_b: M = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4 \longrightarrow W = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2 \tag{6.13}$$

is the basic fibration of $\mathcal{F}$.

Finally, suppose that $\alpha \in \mathbb{R} - \mathbb{Q}$ is an irrational number. Then $K = (\mathbb{Z} \times \mathbb{R}^2, \cdot) \subset G$, $K\backslash G = \mathbb{S}^1$, $\pi_b = \pi_S$: $M = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4 \to W = \mathbb{S}^1$ is the basic fibration of $\mathcal{F}$, the leaves of $\mathcal{F}$ are diffeomorphic to $\mathbb{R}^2$, $\mathcal{F}_b = \mathcal{F}_{\mathfrak{h}}$ is the basic foliation, and $\mathcal{C}(\mathcal{F}) = Q_{\mathfrak{h}}$ is the Molino commuting sheaf of $\mathcal{F}$. Now, to compute E_i and $E_i(\nabla)$, we need to use the following.

Definition 6.14: Let $\alpha \in \mathbb{R} - \mathbb{Q}$ be an irrational number, and consider the vector $v = (1, \alpha)$ in the Euclidean space $\mathbb{R}^2 = (\mathbb{R}^2, \langle , \rangle)$. One says that α satisfies a **Diophantine condition** if there exist positive constants C and δ such that

$$|\langle m, v\rangle| \geq C / \|m\|^\delta \qquad \text{for all } m \in \mathbb{Z}^2 - \{0\}, \tag{6.15}$$

so that

$$(6.16)\qquad \begin{aligned} &|\langle m, v_3\rangle| \geq C/\|m\|^\delta && \text{for all } m \in \mathbb{Z}^2 \times \{0\} - \{0\},\\ &|\langle m, v_4\rangle| \geq C/\|m\|^\delta && \text{for all } m \in \mathbb{Z}^2 \times (\mathbb{Z}^2 - \{0\}). \end{aligned}$$

Otherwise, α is called a **Liouville number**. Hence, if α is a Liouville number, then there exists a sequence $\{m_s\}_{s\geq 1}$ of elements of $\mathbb{Z}^2 \times \{0\} - \{0\}$ such that

$$(6.17)\qquad \begin{aligned} &0 < |\langle m_s, v_3\rangle| < 1/\|m_s\|^s \quad \text{for all } s = 1, 2, \ldots,\\ &m_s \neq m_{s'} \quad \text{and} \quad m_s \neq -m_{s'} \quad \text{if } s \neq s'. \end{aligned}$$

Then we have the following result.

THEOREM 6.18: *Let the situation be as above. Then we have:*

(i) *If $\alpha \in \mathbb{Q}$, then*

$$E_1^{u,v} = \Lambda^u(\omega_0, \omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4) \otimes A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2),$$

with the C^∞-Fréchet topology, for $0 \leq u \leq 3$, $0 \leq v \leq 2$.

(ii) *If $\alpha \in \mathbb{R} - \mathbb{Q}$ satisfies a Diophantine condition, then*

$$E_1^{u,v} = \Lambda^u(\omega_0, \omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4) \otimes A^0(\mathbb{S}^1),$$

with the C^∞-Fréchet topology, for $0 \leq u \leq 3$, $0 \leq v \leq 2$.

(iii) *If $\alpha \in \mathbb{R} - \mathbb{Q}$ is a Liouville number, then $E_1 = \bar{O} \oplus \mathbb{E}_1$ as topological complexes, $E_1^{\cdot,0} = \mathbb{E}_1^{\cdot,0}$, $E_1^{\cdot,v}$ is not Hausdorff and $\bar{O}^{\cdot,v}$ is infinite dimensional for each $v = 1, 2$, and*

$$\mathbb{E}_1^{u,v} = \Lambda^u(\omega_0, \omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4) \otimes A^0(\mathbb{S}^1),$$

with the C^∞-Fréchet topology, for $0 \leq u \leq 3$, $0 \leq v \leq 2$.

(iv) *For any $\alpha \in \mathbb{R}$, the spectral sequence E_i collapses at the second term, and $E_2 = H(M)$ is given by*

$$\begin{aligned} &E_2^{0,0} = E_2^{3,0} = E_2^{0,1} = E_2^{3,1} = E_2^{0,2} = E_2^{3,2} = \mathbb{R},\\ &E_2^{1,0} = E_2^{2,0} = E_2^{1,2} = E_2^{2,2} = \mathbb{R}^2, \quad E_2^{1,1} = E_2^{2,1} = \mathbb{R}^3. \end{aligned}$$

(v) *For any $\alpha \in \mathbb{R}$, the spectral sequence $E_i(\nabla)$ collapses at the second term, and $E_2(\nabla) = H(M, Q_{\mathfrak{h}})$ is given by*

$$\begin{aligned} &E_2^{0,0}(\nabla) = E_2^{3,0}(\nabla) = E_2^{0,2}(\nabla) = E_2^{3,2}(\nabla) = \mathbb{R}, \ E_2^{0,1}(\nabla) = E_2^{3,1}(\nabla) = \mathbb{R}^2,\\ &E_2^{1,0}(\nabla) = E_2^{2,0}(\nabla) = E_2^{1,2}(\nabla) = E_2^{2,2}(\nabla) = \mathbb{R}^3, \ E_2^{1,1}(\nabla) = E_2^{2,1}(\nabla) = \mathbb{R}^5. \end{aligned}$$

Proof: First, we bigrade $A(M)$ by setting

$$A^{u,v}(M) = E_0^{u,v} = \Lambda^u(\omega_0, \omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4) \otimes A^0(M), \quad 0 \le u \le 3, 0 \le v \le 2.$$

It follows from (6.8) that $d_{\mathcal{F}}\omega_i = 0$, $i = 0,1,2,3,4$. Then we have $E_1^{u,v} = \Lambda^u(\omega_0, \omega_1, \omega_2) \otimes E_1^{0,v}$. Clearly, $E_1^{0,0} = A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2)$ if $\alpha \in \mathbb{Q}$, and $E_1^{0,0} = A^0(\mathbb{S}^1)$ if $\alpha \in \mathbb{R} - \mathbb{Q}$. Therefore, to prove (i), (ii) and (iii) it suffices to compute $E_1^{0,1}$ and $E_1^{0,2}$; that is, we need to compute the maps

$$A^0(M) \xrightarrow{d_{\mathcal{F}}} A^{0,1}(M) = (\omega_3, \omega_4) \otimes A^0(M) \xrightarrow{d_{\mathcal{F}}} A^{0,2}(M) = (\omega_3 \wedge \omega_4) \otimes A^0(M).$$

Note that the elements $f \in A^0(M)$ are the smooth functions $f\colon \mathbb{R} \times \mathbb{T}^4 \to \mathbb{R}$ such that

$$f(t+1, A(x)) = f(t,x), \qquad t \in \mathbb{R}, \quad x \in \mathbb{T}^4. \tag{6.19}$$

For each $m = (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4$, denote by e_m the smooth function e_m: $\mathbb{T}^4 \to \mathbb{C}$ given by $e_m(x) = e^{2\pi i \langle m, x\rangle}$. Then, for each $t \in \mathbb{R}$, the Fourier series expansion of $f \in A^0(M)$ is given by

$$f = \sum_{m \in \mathbb{Z}^4} f_m(t) e_m, \tag{6.20}$$

where $f_m\colon \mathbb{R} \to \mathbb{C}$ is a smooth function for all $m \in \mathbb{Z}^4$. It is easy to see that formula (6.19) is equivalent to the formula

$$f_m(t+1) = f_{A'(m)}(t), \qquad m \in \mathbb{Z}^4, \quad t \in \mathbb{R}, \tag{6.21}$$

where A' is the transpose matrix of A. In particular, we have $f_m(t+1) = f_m(t)$ for all $m \in \mathbb{Z}^2 \times \{0\}$ and $t \in \mathbb{R}$, so that

$$f_m\colon \mathbb{S}^1 \longrightarrow \mathbb{C} \text{ is a smooth function for all } m \in \mathbb{Z}^2 \times \{0\}, \text{ and } f_0 \in A^0(\mathbb{S}^1). \tag{6.22}$$

It is clear that $d_{\mathcal{F}} f = X_3(f)\omega_3 + X_4(f)\omega_4 \in A^{0,1}(M)$, and that

$$\begin{aligned} X_3(f) &= 2\pi i \textstyle\sum_{m \in \mathbb{Z}^4} (\langle m, v_3\rangle + t\langle m, v_4\rangle) f_m(t) e_m \in A^0(M), \\ X_4(f) &= 2\pi i \textstyle\sum_{m \in \mathbb{Z}^4} \langle m, v_4\rangle f_m(t) e_m \in A^0(M). \end{aligned} \tag{6.23}$$

Similarly, for $\varphi = g\omega_3 + h\omega_4 \in A^{0,1}(M)$ with $g, h \in A^0(M)$, we have $d_{\mathcal{F}}\varphi = (X_3(h) - X_4(g))\omega_3 \wedge \omega_4 \in A^{0,2}(M)$ and

$$X_3(h) - X_4(g) = 2\pi i \sum_{m \in \mathbb{Z}^4} \left((\langle m, v_3\rangle + t\langle m, v_4\rangle) h_m(t) - \langle m, v_4\rangle g_m(t) \right) e_m, \tag{6.24}$$

where $g_m(t)$ and $h_m(t)$ are the corresponding Fourier coefficients of g and h. Hence, $\varphi \in \operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M)$ if and only if

$$\langle m, v_4\rangle g_m(t) = (\langle m, v_3\rangle + t\langle m, v_4\rangle) h_m(t), \quad m \in \mathbb{Z}^4,\ t \in \mathbb{R}, \tag{6.25}$$

so that $h_m = 0$ if $\langle m, v_3\rangle \neq 0$ and $\langle m, v_4\rangle = 0$.

To prove (i), suppose that $\alpha \in \mathbb{Q}$, so that $\alpha = a_0/a$ with $a_0, a \in \mathbb{Z}$, $a > 0$ and a_0, a relatively prime. Consider the set

$$\mathbb{Z}^2_\alpha = \left\{m \in \mathbb{Z}^4 \mid \langle m, v_3\rangle = \langle m, v_4\rangle = 0\right\}.$$

Evidently, $\mathbb{Z}^2_\alpha \subset \mathbb{Z}^4$ is an additive subgroup and the map

$$\mathbb{Z}^2 \xrightarrow{\cong} \mathbb{Z}^2_\alpha \quad \text{given by} \quad (m_1, m_2) \longmapsto (-a_0 m_1, a m_1, -a_0 m_2, a m_2)$$

is an isomorphism of additive groups. Then, for $f \in A^0(M)$, it follows from (6.12) and (6.13) that

$$f \in E_1^{0,0} = A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2) \overset{\pi_b^*}{\hookrightarrow} A^0(M) \Longleftrightarrow f = \sum_{m \in \mathbb{Z}^2_\alpha} f_m(t) e_m;$$

that is, $f_m = 0$ for all $m \in \mathbb{Z}^4 - \mathbb{Z}^2_\alpha$.

Now, to compute $E_1^{0,1}$, let $\varphi \in \operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M)$ be as above. For each $m \in \mathbb{Z}^4$, consider the smooth function $f_m\colon \mathbb{R} \to \mathbb{C}$ given by

$$f_m(t) = \begin{cases} (2\pi i\langle m, v_3\rangle)^{-1} g_m(t) & \text{if } \langle m, v_3\rangle \neq 0 \text{ and } \langle m, v_4\rangle = 0, \\ (2\pi i\langle m, v_4\rangle)^{-1} h_m(t) & \text{if } \langle m, v_4\rangle \neq 0, \\ 0 & \text{if } \langle m, v_3\rangle = \langle m, v_4\rangle = 0. \end{cases} \tag{6.26}$$

It is clear that $f_m(t+1) = f_{A'(m)}(t)$ for all $m \in \mathbb{Z}^4$ and $t \in \mathbb{R}$. Let

$$C = \min\left(\min_{\langle m, v_3\rangle \neq 0} |\langle m, v_3\rangle|,\ \min_{\langle m, v_4\rangle \neq 0} |\langle m, v_4\rangle|\right) > 0.$$

Then, for any nonnegative integers r, s, it follows from (6.26) that

$$\sum_{m \in \mathbb{Z}^4} \|m\|^r \left|\frac{d^s f_m(t)}{dt^s}\right| \leq$$

$$(2\pi C)^{-1}\left(\sum_{\substack{\langle m, v_3\rangle \neq 0 \\ \langle m, v_4\rangle = 0}} \|m\|^r \left|\frac{d^s g_m(t)}{dt^s}\right| + \sum_{\langle m, v_4\rangle \neq 0} \|m\|^r \left|\frac{d^s h_m(t)}{dt^s}\right|\right).$$

Since g and h are smooth functions, the series on the right converge uniformly on any compact subset of $\mathbb{R}$, so that the series on the left satisfies this property.

Therefore, the $f_m(t)$ are the Fourier coefficients of a smooth real valued function f on M. It follows from (6.23), (6.25) and (6.26) that

$$\varphi = d_{\mathcal{F}} f + (\tilde{g}\omega_3 + \tilde{h}\omega_4) \quad \text{with } \tilde{g} = \sum_{m \in \mathbb{Z}^2_\alpha} g_m(t) e_m, \quad \tilde{h} = \sum_{m \in \mathbb{Z}^2_\alpha} h_m(t) e_m.$$

Clearly, $\tilde{g}, \tilde{h} \in A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2)$. Thus we have

$$\operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M) = d_{\mathcal{F}}(A^0(M)) \oplus (\omega_3, \omega_4) \otimes A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2)$$

with the $\mathcal{C}^\infty$-Fréchet topology, so that

$$E_1^{0,1} = (\omega_3, \omega_4) \otimes A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2)$$

with the $\mathcal{C}^\infty$-Fréchet topology.

On the other hand, to compute $E_1^{0,2}$, consider an element $\psi = f\omega_3 \wedge \omega_4 \in A^{0,2}(M)$, and let $f_m(t)$ be the Fourier coefficients of $f \in A^0(M)$. Then we have an element $\varphi = g\omega_3 + h\omega_4 \in A^{0,1}(M)$ such that the corresponding Fourier coefficients $g_m(t)$ and $h_m(t)$ of $g, h \in A^0(M)$ are given by

$$\text{(6.27)} \quad \begin{aligned} g_m(t) &= \begin{cases} -(2\pi i \langle m, v_4 \rangle)^{-1} f_m(t) & \text{if } \langle m, v_4 \rangle \neq 0, \\ 0 & \text{otherwise}; \end{cases} \\ h_m(t) &= \begin{cases} (2\pi i \langle m, v_3 \rangle)^{-1} f_m(t) & \text{if } \langle m, v_3 \rangle \neq 0 \quad \text{and } \langle m, v_4 \rangle = 0, \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

It follows from (6.24) and (6.27) that

$$\psi = d_{\mathcal{F}}\varphi + \tilde{f}\omega_3 \wedge \omega_4 \quad \text{with } \tilde{f} = \sum_{m \in \mathbb{Z}^2_\alpha} f_m(t) e_m \in A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2).$$

Therefore we have

$$A^{0,2}(M) = d_{\mathcal{F}}(A^{0,1}(M)) \oplus (\omega_3 \wedge \omega_4) \otimes A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2)$$

with the $\mathcal{F}^\infty$-Fréchet topology, so that

$$E_1^{0,2} = (\omega_3 \wedge \omega_4) \otimes A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2)$$

with the $\mathcal{F}^\infty$-Fréchet topology. This completes the proof of part (i).

Next, suppose that $\alpha \in \mathbb{R} - \mathbb{Q}$. Then, for any $m \in \mathbb{Z}^4$, we have the following relations:

$$\text{(6.28)} \quad \begin{aligned} &\langle m, v_3 \rangle \neq 0 \text{ and } \langle m, v_4 \rangle = 0 \Longleftrightarrow m \in \mathbb{Z}^2 \times \{0\} - \{0\}, \\ &\langle m, v_4 \rangle \neq 0 \Longleftrightarrow m \in \mathbb{Z}^2 \times (\mathbb{Z}^2 - \{0\}), \\ &\langle m, v_3 \rangle = \langle m, v_4 \rangle = 0 \Longleftrightarrow m = 0. \end{aligned}$$

To prove (ii), assume that $\alpha \in \mathbb{R} - \mathbb{Q}$ satisfies a Diophantine condition. Let $\varphi \in \operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M)$ be as above. Then, for each $m \in \mathbb{Z}^4$, consider the smooth function $f_m \colon \mathbb{R} \to \mathbb{C}$ given by (6.26). Now, for any nonnegative integers r, s, it follows from (6.16) and (6.26) that

$$\sum_{m\in\mathbb{Z}^4} \|m\|^r \left|\frac{d^s f_m(t)}{dt^s}\right| \leq$$
$$\tilde{C}\left(\sum_{m\in\mathbb{Z}^2\times\{0\}-\{0\}} \|m\|^{r+\delta}\left|\frac{d^s g_m(t)}{dt^s}\right| + \sum_{m\in\mathbb{Z}^2\times(\mathbb{Z}^2-\{0\})} \|m\|^{r+\delta}\left|\frac{d^s h_m(t)}{dt^s}\right|\right),$$

where $\tilde{C} = (2\pi C)^{-1} > 0$. Hence, the $f_m(t)$ are the Fourier coefficients of a smooth real valued function f on M such that

$$\varphi = d_{\mathcal{F}} f + (g_0\omega_3 + h_0\omega_4) \qquad \text{with } g_0, h_0 \in A^0(\mathbb{S}^1).$$

This implies that

$$\operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M) = d_{\mathcal{F}}(A^0(M)) \oplus (\omega_3, \omega_4) \otimes A^0(\mathbb{S}^1)$$

with the C^∞-Fréchet topology, so that

$$E_1^{0,1} = (\omega_3, \omega_4) \otimes A^0(\mathbb{S}^1)$$

with the C^∞-Fréchet topology. Similarly, using (6.16) and (6.27) we obtain

$$A^{0,2}(M) = d_{\mathcal{F}}(A^{0,1}(M)) \oplus (\omega_3 \wedge \omega_4) \otimes A^0(\mathbb{S}^1)$$

with the C^∞-Fréchet topology. Thus

$$E_1^{0,2} = (\omega_3 \wedge \omega_4) \otimes A^0(\mathbb{S}^1)$$

with the C^∞-Fréchet topology. This proves (ii).

To prove (iii), suppose that $\mathbb{R} - \mathbb{Q}$ is a Liouville number. Then we have

$$\operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M) = D^{0,1}(M) \oplus (\omega_3, \omega_4) \otimes A^0(\mathbb{S}^1) \tag{6.29}$$

with the C^∞-Fréchet topology, where $D^{0,1}(M)$ is the closed subspace of $\operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M)$ given by

$$D^{0,1}(M) = \{\varphi = g\omega_3 + h\omega_4 \in \operatorname{Ker} d_{\mathcal{F}} \cap A^{0,1}(M) \mid g_0 = h_0 = 0\}.$$

It is clear that $d_{\mathcal{F}}(A^0(M)) \subset D^{0,1}(M)$. Now, to prove that $\overline{d_{\mathcal{F}}(A^0(M))} = D^{0,1}(M)$, consider an element $\varphi = g\omega_3 + h\omega_4 \in D^{0,1}(M)$, and let $g_m(t)$ and

$h_m(t)$ be the corresponding Fourier coefficients of $g, h \in A^0(M)$. Let $\{f_k\}_{k\geq 1}$ be the sequence of elements of $A^0(M)$ given by

$$f_k = \sum_{\substack{m\in\mathbb{Z}^2\times\{0\}-\{0\}\\ \|m\|\leq k}} \frac{g_m(t)}{2\pi i\langle m, v_3\rangle} e_m + \sum_{\substack{m\in\mathbb{Z}^2\times(\mathbb{Z}^2-\{0\})\\ \|(m_3,m_4)\|\leq k}} \frac{h_m(t)}{2\pi i\langle m, v_4\rangle} e_m.$$

Then we have a sequence $\{d_{\mathcal{F}} f_k\}_{k\geq 1}$ of elements of $d_{\mathcal{F}}(A^0(M))$, which converges to φ, so that $\overline{d_{\mathcal{F}}(A^0(M))} = D^{0,1}(M)$. Now, using (6.29) we obtain the topological identities

$$E_1^{0,1} = \bar{O}^{0,1} \oplus \mathbb{E}_1^{0,1}, \quad \bar{O}^{0,1} = D^{0,1}(M)/d_{\mathcal{F}}(A^0(M)), \quad \mathbb{E}_1^{0,1} = (\omega_3, \omega_4) \otimes A^0(\mathbb{S}^1).$$

To prove that $\bar{O}^{0,1}$ is infinite dimensional, consider a sequence $\{m_s\}_{s\geq 1}$ of elements of $\mathbb{Z}^2\times\{0\}-\{0\}$ satisfying the conditions (6.17). For each $\lambda \in [0,\infty) \subset \mathbb{R}$, let $\varphi_\lambda = g_\lambda\omega_3 + h_\lambda\omega_4 \in D^{0,1}(M)$ be the element such that $h_\lambda = 0$ and the Fourier coefficients $(g_\lambda)_m \in \mathbb{R}$ of $g_\lambda \in A^0(M)$ are given by

$$(6.30) \qquad (g_\lambda)_m = \begin{cases} \|m_s\|^{-s/2} s^\lambda & \text{if } m = m_s \text{ or if } m = -m_s, \\ 0 & \text{otherwise.} \end{cases}$$

To show that the set $\{[\varphi_\lambda]\}_{\lambda\in[0,\infty)} \subset \bar{O}^{0,1}$ is linearly independent, consider a linear combination

$$\sum_{j=1}^{r} a_j \left[\varphi_{\lambda_j}\right] = 0, \qquad a_j \in \mathbb{R}, \quad 0 \leq \lambda_1 < \cdots < \lambda_r < \infty.$$

Then there exists an element $f \in A^0(M)$ such that

$$(6.31) \qquad d_{\mathcal{F}} f = \sum_{j=1}^{r} a_j \varphi_{\lambda_j}.$$

Let f_m be the Fourier coefficients of f. It follows from (6.23), (6.30) and (6.31) that

$$f_{m_s} = (2\pi i\langle m_s, v_3\rangle)^{-1} \|m_s\|^{-s/2} \sum_{j=1}^{r} a_j s^{\lambda_j}, \qquad s = 1, 2, \ldots.$$

Then by (6.17) we have

$$|f_{m_s}| = (2\pi\,|\langle m_s, v_3\rangle|)^{-1} \|m_s\|^{-s/2} \left|\sum_{j=1}^{r} a_j s^{\lambda_j}\right| \geq (2\pi)^{-1} \|m_s\|^{s/2} \left|\sum_{j=1}^{r} a_j s^{\lambda_j}\right|$$

for all $s \geq 1$. Now, since

$$\lim_{s\to\infty} |f_{m_s}| = 0 \qquad \text{and} \qquad \lim_{s\to\infty} \|m_s\|^{s/2} = \infty,$$

we have

$$\lim_{s\to\infty}\left|\sum_{j=1}^{r} a_j s^{\lambda_j}\right| = 0.$$

Therefore, using the relation $0 \le \lambda_1 < \cdots < \lambda_r < \infty$, we obtain $a_j = 0$ for all $j = 1, \dots, r$. This proves that $\bar{O}^{0,1}$ is infinite dimensional.

To compute $E_1^{0,2}$, consider the topological identity

$$A^{0,2} = D^{0,2}(M) \oplus (\omega_3 \wedge \omega_4) \otimes A^0(\mathbb{S}^1),$$

where $D^{0,2}(M)$ is the closed subspace of $A^{0,2}(M)$ given by

$$D^{0,2}(M) = \{\psi = f\omega_3 \wedge \omega_4 \in A^{0,2}(M) \mid f_0 = 0\}.$$

Evidently, $d_{\mathcal{F}}(A^{0,1}(M)) \subset D^{0,2}(M)$. Let $\psi = f\omega_3 \wedge \omega_4 \in D^{0,2}(M)$ be an element, and let $f_m(t)$ be the Fourier coefficients of $f \in A^0(M)$. Then we obtain a sequence $\{d_{\mathcal{F}}\varphi_k\}_{k\ge1}$ of elements of $d_{\mathcal{F}}(A^{0,1}(M))$, which converges to ψ, where $\varphi_k = g_k\omega_3 + h_k\omega_4 \in A^{0,1}(M)$ with $k \ge 1$, and $g_k, h_k \in A^0(M)$ are given by

$$g_k = -\sum_{\substack{m\in\mathbb{Z}^2\times(\mathbb{Z}^2-\{0\})\\ \|(m_3,m_4)\|\le k}} \frac{f_m(t)}{2\pi i\langle m, v_4\rangle} e_m, \qquad h_k = \sum_{\substack{m\in\mathbb{Z}^2\times\{0\}-\{0\}\\ \|m\|\le k}} \frac{f_m(t)}{2\pi i\langle m, v_3\rangle} e_m.$$

This shows that $\overline{d_{\mathcal{F}}(A^{0,1}(M))} = D^{0,2}(M)$. Hence we have the topological identities

$$E_1^{0,2} = \bar{O}^{0,2} \oplus \mathbb{E}_1^{0,2}, \quad \bar{O}^{0,2} = D^{0,2}(M)/d_{\mathcal{F}}(A^{0,1}(M)), \quad \mathbb{E}_1^{0,2} = (\omega_3 \wedge \omega_4) \otimes A^0(\mathbb{S}^1).$$

To prove that $\bar{O}^{0,2}$ is infinite dimensional, consider the subset $\{[\psi_\lambda]\}_{\lambda\in[0,\infty)}$ of $\bar{O}^{0,2}$ such that $\psi_\lambda = f_\lambda\omega_3 \wedge \omega_4 \in D^{0,2}(M)$, $\lambda \ge 0$, and $f_\lambda \in A^0(M)$ is the smooth function, whose Fourier coefficients $(f_\lambda)_m$ are given by (6.30). Now, suppose that

$$d_{\mathcal{F}}\varphi = \sum_{j=1}^{r} a_j \psi_{\lambda_j}, \quad a_j \in \mathbb{R}, \quad 0 \le \lambda_1 < \cdots < \lambda_r < \infty,$$

where $\varphi = g\omega_3 + h\omega_4 \in A^{0,1}(M)$ with $g, h \in A^0(M)$. Then for each $s = 1, 2, \dots$, the Fourier coefficient h_{m_s} of h is given by

$$h_{m_s} = (2\pi i\langle m_s, v_3\rangle)^{-1} \|m_s\|^{-s/2} \sum_{j=1}^{r} a_j s^{\lambda_j}.$$

It follows that $a_j = 0$ for all $j = 1, \dots, r$. This shows that the set $\{[\psi_\lambda]\}_{\lambda\in[0,\infty)} \subset \bar{O}^{0,2}$ is linearly independent. Thus $\bar{O}^{0,2}$ is infinite dimensional. Hence, part (iii) is completely proved.

Finally, parts (iv) and (v) follow immediatly from parts (i), (ii) and (iii), and Theorems 5.3 and 5.4. This completes the proof of the theorem. ∎

For each $t \in \mathbb{R}$, let $\mathcal{F}_t$ (resp. $\mathcal{F}_{1,t}$) be the Lie $\mathbb{R}^2$-foliation of dimension 2 induced by $\mathcal{F}$ (resp. by $\mathcal{F}_1$) on the fiber $\mathbb{T}^4_t = \pi_S^{-1}(\pi_S(t))$ of π_S: $M = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4 \to \mathbb{S}^1$, which is canonically globally isomorphic to the $\mathbb{R}^2$-foliation on $\mathbb{T}^4$ defined by v_3, v_4 (resp. by v_1, v_2). Consider for example $\mathcal{F}_t$, since the same argument applies to $\mathcal{F}_{1,t}$. Clearly, if $\alpha \in \mathbb{Q}$, then the leaves of $\mathcal{F}_t$ are the fibers of the basic fibration $(\pi_b)_t$: $\mathbb{T}^4_t \equiv \mathbb{T}^4 \to \mathbb{T}^2_t \equiv \mathbb{T}^2$ with fiber $\mathbb{T}^2$. Similarly, if $\alpha \in \mathbb{R} - \mathbb{Q}$, then the leaves of $\mathcal{F}_t$ are dense in $\mathbb{T}^4_t$. Then, using the proof of Theorem 6.18, we obtain the following result.

PROPOSITION 6.32: *For each $t \in \mathbb{R}$, the spectral sequence (E_i, d_i) associated to $\mathcal{F}_t$ satisfies the following properties:*

(i) *If $\alpha \in \mathbb{Q}$, then $E_1^{u,v} = \Lambda^u(\omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4) \otimes A^0(\mathbb{T}^2)$, with the C^∞-Fréchet topology, for $0 \le u \le 2$, $0 \le v \le 2$.*

(ii) *If $\alpha \in \mathbb{R} - \mathbb{Q}$ satisfies a Diophantine condition, then $E_1^{u,v} = E_2^{u,v} = \Lambda^u(\omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4)$, with the C^∞-Fréchet topology, for $0 \le u \le 2$, $0 \le v \le 2$.*

(iii) *If $\alpha \in \mathbb{R} - \mathbb{Q}$ is a Liouville number, then $E_1 = \bar{O} \oplus \mathbb{E}_1$ as topological complexes, $E_1^{\cdot,0} = \mathbb{E}_1^{\cdot,0}$, $E_1^{\cdot,v}$ is not Hausdorff and $\bar{O}^{\cdot,v}$ is infinite dimensional for each $v = 1, 2$, and $\mathbb{E}_1^{u,v} = E_2^{u,v} = \Lambda^u(\omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4)$, with the C^∞-Fréchet topology, for $0 \le u \le 2$, $0 \le v \le 2$.*

(iv) *For any $\alpha \in \mathbb{R}$, the spectral sequence E_i collapses at the second term, and $E_2 = H(\mathbb{T}^4)$ is given by $E_2^{u,v} = \Lambda^u(\omega_1, \omega_2) \otimes \Lambda^v(\omega_3, \omega_4) \cong \mathbb{R}^{u+v}$ for $0 \le u \le 2$, $0 \le v \le 2$.*

Example 2: Let A be the matrix in $\mathrm{SL}(4, \mathbb{Z})$ given by

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right), \qquad \text{where } A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad A_2 = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$$

is a matrix in $\mathrm{SL}(2, \mathbb{Z})$ such that $\mathrm{tr}(A_2) > 2$. Then $a_2 a_3 \neq 0$ and A_2 has two positive irrational real eigenvalues λ and λ^{-1}. Let $\alpha = (\lambda - a_1)/a_3 \in \mathbb{R} - \mathbb{Q}$, so that $(-(a_3/a_2)\alpha, 1)$, $(1, \alpha)$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors associated to λ^{-1} and λ respectively. It follows that $A_2^t \in \mathrm{SL}(2, \mathbb{R})$ for all $t \in \mathbb{R}$. Therefore we have

$$A^t = \left(\begin{array}{c|c} A_1^t & 0 \\ \hline 0 & A_2^t \end{array}\right) \in \mathrm{SL}(4, \mathbb{R}) \qquad \text{with } A_1^t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \qquad \text{for all } t \in \mathbb{R}.$$

Now, let $G_A = \mathbb{R} \times_\phi \mathbb{R}^4$ be the semidirect product of the additive Lie groups $\mathbb{R}$ and $\mathbb{R}^4$ via the representation $\phi\colon \mathbb{R} \to \mathrm{SL}(4,\mathbb{R})$ defined by $\phi(t) = A^t$; that is, $G_A = (\mathbb{R}^5, \cdot)$ with the group operation given by

$$\begin{aligned}&(t, x_1, x_2, x_3, x_4) \cdot (t', x_1', x_2', x_3', x_4')\\ =&(t+t', (x_1, x_2, x_3, x_4) + A^t(x_1', x_2', x_3', x_4'))\\ =&(t+t', x_1 + x_1', x_2 + x_2' + tx_1', (x_3, x_4) + A_2^t(x_3', x_4')).\end{aligned}$$

So we have constructed a simply connected solvable Lie group G_A of dimension 5, which is not nilpotent. Clearly, $\Gamma_A = (\mathbb{Z}^5, \cdot) \subset G_A$ is a discrete uniform and torsion-free subgroup, and the compact connected homogeneous space $M = \Gamma_A \backslash G_A$ of dimension 5 is the quotient manifold $\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4$ of $\mathbb{R} \times \mathbb{T}^4$ by the equivalence relation given by $(t, x) \sim (t+1, A(x))$, $t \in \mathbb{R}$, $x \in \mathbb{T}^4$, where A also denotes the automorphism of $\mathbb{T}^4$ induced by A. Moreover, the canonical projection $\pi_S\colon M = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4 \to \mathbb{S}^1$ is a flat bundle with fiber $\mathbb{T}^4$.

On the other hand, consider the basis of the Euclidean space $\mathbb{R}^4 \equiv (\mathbb{R}^4, \langle , \rangle)$ given by

$$v_1 = (0, 0, -(a_3/a_2)\alpha, 1), \quad v_2 = (1, 0, 0, 0), \quad v_3 = (0, 1, 0, 0), \quad v_4 = (0, 0, 1, \alpha),$$

which satisfies the identities

$$A^t(v_1) = \lambda^{-t} v_1, \ A^t(v_2) = v_2 + tv_3, \ A^t(v_3) = v_3, \ A^t(v_4) = \lambda^t v_4 \ \text{ for all } t \in \mathbb{R}.$$

Note that $-(a_3/a_2)\alpha, \alpha \in \mathbb{R} - \mathbb{Q}$ are algebraic numbers over $\mathbb{Q}$, so that they satisfy Diophantine conditions. Hence, there exist positive constants C and δ such that

$$(6.33)\ |\langle m, v_1\rangle| \geq C/\|m\|^\delta, |\langle m, v_4\rangle| \geq C/\|m\|^\delta \quad \text{for all } m \in \mathbb{Z}^2 \times (\mathbb{Z}^2 - \{0\}).$$

Now, consider the basis of left invariant vector fields on G_A given by

$$\begin{aligned}X_0 =&\frac{\partial}{\partial t}, \quad X_1 = \lambda^{-t}\left(-(a_3/a_2)\alpha\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right),\\ X_2 =&\frac{\partial}{\partial x_1} + t\frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_2}, \quad X_4 = \lambda^t\left(\frac{\partial}{\partial x_3} + \alpha\frac{\partial}{\partial x_4}\right),\end{aligned}$$

where X_i is induced by v_i, $i = 1,2,3,4$. For each $i = 0,1,2,3,4$, X_i defines a vector field, also denoted X_i, on $M = \Gamma_A \backslash G_A = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4$, and $X_0, X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$ is a parallelism on M satisfying

$$(6.34)\quad \begin{aligned}&[X_0, X_1] = (-\log\lambda)X_1, \quad [X_0, X_2] = X_3, \quad [X_0, X_4] = (\log\lambda)X_4,\\ &[X_i, X_j] = 0 \qquad \text{otherwise.}\end{aligned}$$

Then the dual basis $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4 \in A^1(M)$ of X_0, X_1, X_2, X_3, X_4 is given by

$$\omega_0 = dt, \quad \omega_1 = c\lambda^t(-\alpha dx_3 + dx_4), \quad \omega_2 = dx_1,$$
$$\omega_3 = -tdx_1 + dx_2, \quad \omega_4 = c\lambda^{-t}(dx_3 + (a_3/a_2)\alpha dx_4),$$

where $c = 1/(1 + (a_3/a_2)\alpha^2)$. Therefore we have

$$\begin{aligned} d\omega_1 &= (\log\lambda)\omega_0 \wedge \omega_1, \quad d\omega_3 = -\omega_0 \wedge \omega_2, \\ d\omega_4 &= (-\log\lambda)\omega_0 \wedge \omega_4, \quad d\omega_0 = d\omega_2 = 0. \end{aligned} \tag{6.35}$$

Evidently, the elements $f \in A^0(M)$ are the smooth functions $f\colon \mathbb{R} \times \mathbb{T}^4 \to \mathbb{R}$, whose Fourier coefficients $f_m\colon \mathbb{R} \to \mathbb{C}$, $m \in \mathbb{Z}^4$, satisfy

$$f_m(t+1) = f_{A'(m)}(t), \quad m \in \mathbb{Z}^4, \quad t \in \mathbb{R}, \tag{6.36}$$

where A' is the transpose matrix of A. In particular, we have $f_m(t+1) = f_m(t)$ for all $m \in \mathbb{Z} \times \{0\}$ and $t \in \mathbb{R}$. Hence, $f_m\colon \mathbb{S}^1 \to \mathbb{C}$ is a smooth function for all $m \in \mathbb{Z} \times \{0\}$, and $f_0 \in A^0(\mathbb{S}^1)$. Furthermore, for any $m \in \mathbb{Z}^4$, we have the following relations:

$$\begin{aligned} &\langle m, v_2\rangle = \langle m, v_3\rangle = \langle m, v_4\rangle = 0 \iff m = 0, \\ &\langle m, v_3\rangle = \langle m, v_4\rangle = 0 \iff A'(m) = m \iff m \in \mathbb{Z} \times \{0\}, \\ &\langle m, v_2\rangle \neq 0 \text{ and } \langle m, v_3\rangle = \langle m, v_4\rangle = 0 \iff m \in \mathbb{Z} \times \{0\} - \{0\}, \\ &\langle m, v_3\rangle \neq 0 \text{ and } \langle m, v_4\rangle = 0 \iff m \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) \times \{0\}, \\ &\langle m, v_4\rangle \neq 0 \iff m \in \mathbb{Z}^2 \times (\mathbb{Z}^2 - \{0\}). \end{aligned} \tag{6.37}$$

Next, by (6.34) it follows that X_4 (resp. X_1) defines a homogeneous Lie flow $\mathcal{F}$ (resp. $\mathcal{F}_1$) on M. Consider for example $\mathcal{F}$, since the same techniques can be used for $\mathcal{F}_1$. Then X_0, X_1, X_2, X_3 define a transverse Lie parallelism of $\mathcal{F}$. Now, let $G = (\mathbb{R}^4, \cdot)$ be the group, whose group operation is given by

$$(t, x, y, z) \cdot (t', x', y', z') = (t + t', x + x', y + y' + tx', z + \lambda^{-t}z').$$

G is a simply connected solvable Lie group of dimension 4, which is not nilpotent. Then the surjective homomorphism of Lie groups $D\colon G_A \to G$ given by $D(t, x_1, x_2, x_3, x_4) = (t, x_1, x_2, -\alpha x_3 + x_4)$ is the developing map of $\mathcal{F}$. Hence, $\mathcal{F}$ is a homogeneous Lie $\mathfrak{g}$-flow and $\Gamma = D(\Gamma_A) \subset G$ is its holonomy group, where $\mathfrak{g}$ is the Lie algebra of G. Clearly, $K = \bar{\Gamma} = (\mathbb{Z}^3 \times \mathbb{R}, \cdot) \subset G$, so that $K\backslash G = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2$ is the compact Heisenberg manifold of dimension 3. It follows that the basic fibration $\pi_b\colon M = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4 \to W = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2$ of $\mathcal{F}$, with fiber $\mathbb{T}^2$, is also induced by the surjective homomorphism of Lie groups

$$G_A \longrightarrow G_3 \quad \text{given by} \quad (t, x_1, x_2, x_3, x_4) \longmapsto (t, x_1, x_2),$$

where G_3 is the Heisenberg group of dimension 3. Then X_1, X_4 define the basic foliation $\mathcal{F}_b$ of $\mathcal{F}$, and $\bar{X}_1 \in \Gamma\mathcal{C}(\mathcal{F})$ is a basis of sections for the Molino commuting sheaf $\mathcal{C}(\mathcal{F})$ of $\mathcal{F}$. Similarly, (6.34) implies that the closed one-form $\gamma = (-\log\lambda)\omega_0 \in A^1(\mathbb{S}^1) \subset A^1(M)$ is the connection form of the canonical flat connection ∇ on $\mathcal{C}(\mathcal{F})$ with respect to $\bar{X}_1$, so that $H(M, \mathcal{C}(\mathcal{F})) = H_\gamma(M)$. Consider now the spectral sequence (E_i, d_i) (resp. $(E_i(\nabla), (d_\nabla)_i)$) associated to $\mathcal{F}$ (resp. to $\mathcal{F}$ and $\mathcal{C}(\mathcal{F})$). It follows that

$$E_i(\nabla) = E_i(\gamma) \quad \text{for all } i \geq 0, \quad \text{and} \quad E_i(\nabla) = E_i \quad \text{for } i = 0, 1.$$

Then, using (6.33), (6.35), (6.36) and (6.37), by the same method as in the proof of Theorem 6.18, we obtain the following result.

THEOREM 6.38: *Let the situation be as above. Then we have:*

(i) $E_1^{u,v} = \Lambda^u(\omega_0, \omega_1, \omega_2, \omega_3) \otimes \Lambda^v(\omega_4) \otimes A^0(\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^2)$, *with the C^∞-Fréchet topology, for $0 \leq u \leq 4$, $0 \leq v \leq 1$.*

(ii) *The spectral sequence E_i collapses at the second term, and $E_2 = H(M)$ is given by*

$$E_2^{0,0} = E_2^{3,0} = E_2^{1,1} = E_2^{4,1} = \mathbb{R},$$
$$E_2^{1,0} = E_2^{2,0} = E_2^{2,1} = E_2^{3,1} = \mathbb{R}^2, \quad E_2^{4,0} = E_2^{0,1} = 0.$$

(iii) *The spectral sequence $E_i(\gamma)$ collapses at the second term, and $E_2(\gamma) = H_\gamma(M)$ is given by*

$$E_2^{1,0}(\gamma) = E_2^{4,0}(\gamma) = \mathbb{R}, \quad E_2^{2,0}(\gamma) = E_2^{3,0}(\gamma) = \mathbb{R}^2,$$
$$E_2^{u,v}(\gamma) = 0 \qquad \textit{otherwise}.$$

Example 3: Let the notation be as in Example 2. Then (6.34) implies that X_3, X_4 (resp. X_3, X_1) define a homogeneous Lie foliation $\mathcal{F}$ (resp. $\mathcal{F}_1$) of dimension 2 on M. Consider for example $\mathcal{F}$, since the same argument applies to $\mathcal{F}_1$. It is clear that X_0, X_1, X_2 define a transverse Lie parallelism of $\mathcal{F}$, and that the surjective homomorphism of Lie groups $D: G_A \to G$ given by $D(t, x_1, x_2, x_3, x_4) = (t, x_1, -\alpha x_3 + x_4)$ is the developing map of $\mathcal{F}$, where $G = (\mathbb{R}^3, \cdot)$ with the group operation given by

$$(t, x, y) \cdot (t', x', y') = (t + t', x + x', y + \lambda^{-t} y').$$

It follows that $K = \overline{D(\Gamma_A)} = (\mathbb{Z}^2 \times \mathbb{R}, \cdot) \subset G$, so that $K\backslash G = \mathbb{T}^2$. Thus the basic fibration $\pi_b: M = \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}^4 \to \mathbb{T}^2$ of $\mathcal{F}$, with fiber $\mathbb{T}^3$, is also induced by the surjective homomorphism of Lie groups

$$G_A \longrightarrow (\mathbb{R}^2, +) \quad \text{given by} \quad (t, x_1, x_2, x_3, x_4) \longmapsto (t, x_1).$$

Evidently, X_1, X_3, X_4 define $\mathcal{F}_b$, $\bar{X}_1 \in \Gamma\mathcal{C}(\mathcal{F})$ is a basis of sections for $\mathcal{C}(\mathcal{F})$, and $\gamma = (-\log \lambda)\omega_0$ is the connection form of the canonical flat connection ∇ on $\mathcal{C}(\mathcal{F})$ with respect to $\bar{X}_1$. Therefore, for $H(M, \mathcal{C}(\mathcal{F}))$ and $E_i(\nabla)$, we have

$$H(M,\mathcal{C}(\mathcal{F})) = H_\gamma(M),\ E_i(\nabla) = E_i(\gamma) \text{ for all } i \geq 0, \text{ and } E_i(\nabla) = E_i, i = 0,1.$$

Then, using (6.33), (6.35), (6.36) and (6.37), and the same techniques as in the proof of Theorem 6.18, we obtain the following result.

THEOREM 6.39: *Let the situation be as above. Then , for E_i and $E_i(\gamma)$, we have:*

(i) $E_1^{u,v} = \Lambda^u(\omega_0,\omega_1,\omega_2)\otimes\Lambda^v(\omega_3,\omega_4)\otimes A^0(\mathbb{T}^2)$, *with the C^∞-Fréchet topology, for $0 \leq u \leq 3$, $0 \leq v \leq 2$.*

(ii) *The spectral sequence E_i collapses at the third term, and E_2 and $E_3 = H(M)$ are given by*

$$\begin{array}{l} E_2^{0,0} = E_2^{2,0} = E_2^{0,1} = E_2^{3,1} = E_2^{1,2} = E_2^{3,2} = \mathbb{R}, \\ E_2^{1,0} = E_2^{2,2} = \mathbb{R}^2, E_2^{1,1} = E_2^{2,1} = \mathbb{R}^3, E_2^{3,0} = E_2^{0,2} = 0, \\ E_3^{0,0} = E_3^{3,2} = \mathbb{R}, E_3^{1,0} = E_3^{2,2} = \mathbb{R}^2, E_3^{1,1} = E_3^{2,1} = \mathbb{R}^3, \\ E_3^{u,v} = 0 \qquad \textit{otherwise.} \end{array}$$

(iii) *The spectral sequence $E_i(\gamma)$ collapses at the third term, and $E_2(\gamma)$ and $E_3(\gamma) = H_\gamma(M)$ are given by*

$$\begin{array}{l} E_2^{1,0}(\gamma) = E_2^{3,0}(\gamma) = E_2^{1,1}(\gamma) = E_2^{3,1}(\gamma) = \mathbb{R}, E_2^{2,0}(\gamma) = E_2^{2,1}(\gamma) = \mathbb{R}^2, \\ E_2^{u,v}(\gamma) = 0 \qquad \textit{otherwise,} \\ E_3^{1,0}(\gamma) = E_3^{3,1}(\gamma) = \mathbb{R}, E_3^{2,0}(\gamma) = E_3^{2,1}(\gamma) = \mathbb{R}^2, \\ E_3^{u,v}(\gamma) = 0 \qquad \textit{otherwise.} \end{array}$$

Example 4: Let the notation be as in Example 2. Then, by (6.34) it follows that X_2, X_3, X_4 (resp. X_2, X_3, X_1) define a homogeneous Lie foliation $\mathcal{F}$ (resp. $\mathcal{F}_1$) of dimension 3 on M. Consider for example $\mathcal{F}$, since the same techniques can be used for $\mathcal{F}_1$. It is easy to see that X_0, X_1 define a transverse Lie parallelism of $\mathcal{F}$, and that the surjective homomorphism of Lie groups $D\colon G_A \to GA$ given by $D(t,x_1,x_2,x_3,x_4) = (t, -\alpha x_3 + x_4)$ is the developing map of $\mathcal{F}$, where $GA = (\mathbb{R}^2, \cdot)$ is the affine group with the group operation given by $(t,x)\cdot(t',x') = (t+t', x+\lambda^{-t}x')$. It follows that $K = \overline{D(\Gamma_A)} = (\mathbb{Z}\times\mathbb{R}, \cdot) \subset GA$, so that $K\backslash GA = \mathbb{S}^1$. Therefore, $\pi_b = \pi_S\colon M = \mathbb{R}\times_{\mathbb{Z}}\mathbb{T}^4 \to \mathbb{S}^1$ is the basic fibration of $\mathcal{F}$. Thus X_1, X_2, X_3, X_4 define $\mathcal{F}_b$, $\bar{X}_1 \in \Gamma\mathcal{C}(\mathcal{F})$ is a basis of sections for $\mathcal{C}(\mathcal{F})$, and $\gamma = (-\log\lambda)\omega_0$ is the connection form of the canonical flat connection ∇ on $\mathcal{C}(\mathcal{F})$ with respect to $\bar{X}_1$. Hence, for $H(M,\mathcal{C}(\mathcal{F}))$ and $E_i(\nabla)$, we have

$$H(M,\mathcal{C}(\mathcal{F})) = H_\gamma(M),\ E_i(\nabla) = E_i(\gamma) \text{ for all } i \geq 0, \text{ and } E_i(\nabla) = E_i, i = 0,1.$$

Then, using (6.33), (6.35), (6.36) and (6.37), and the same argument as in the proof of Theorem 6.18, we obtain the following result.

THEOREM 6.40: *Let the situation be as above. Then , for E_i and $E_i(\gamma)$, we have:*

(i) $E_1^{u,v} = \Lambda^u(\omega_0,\omega_1)\otimes\Lambda^v(\omega_2,\omega_3,\omega_4)\otimes A^0(\mathbb{S}^1)$, *with the* C^∞*-Fréchet topology, for* $0 \le u \le 2$, $0 \le v \le 3$.

(ii) *The spectral sequence* E_i *collapses at the second term, and* $E_2 = H(M)$ *is given by*

$$E_2^{0,0} = E_2^{1,0} = E_2^{0,1} = E_2^{2,1} = E_2^{0,2} = E_2^{2,2} = E_2^{1,3} = E_2^{2,3} = \mathbb{R},$$
$$E_2^{1,1} = E_2^{1,2} = \mathbb{R}^2, \quad E_2^{2,0} = E_2^{0,3} = 0.$$

(iii) *The spectral sequence* $E_i(\gamma)$ *collapses at the second term, and* $E_2(\gamma) = H_\gamma(M)$ *is given by*

$$E_2^{1,0}(\gamma) = E_2^{2,0}(\gamma) = E_2^{1,1}(\gamma) = E_2^{2,1}(\gamma) = E_2^{1,2}(\gamma) = E_2^{2,2}(\gamma) = \mathbb{R},$$
$$E_2^{u,v}(\gamma) = 0 \qquad \text{otherwise.}$$

Remark: If in part (v) of Theorem 6.18 we consider the dual flat connection ∇^* of ∇ on the dual flat bundle $Q_{\mathfrak{h}}^*$ of $Q_{\mathfrak{h}}$, then, for the spectral sequence $E_i(\nabla^*)$, we have

$$E_i^{u,v}(\nabla^*) = E_i^{3-u,2-v}(\nabla) \quad \text{for } i \ge 2,\ 0 \le u \le 3,\ 0 \le v \le 2.$$

Similarly, in part (iii) of Theorems 6.38, 6.39 and 6.40, we can consider the spectral sequence $E_i(-\gamma)$. It is easy to check that

$$E_i^{u,v}(-\gamma) = E_i^{5-p-u,p-v}(\gamma) \quad \text{for } i \ge 2,\ 0 \le u \le 5-p,\ 0 \le v \le p,$$

where $p \in \{1,2,3\}$ is the dimension of the corresponding foliation $\mathcal{F}$ on M.

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